Société de Calcul Mathématique SA
Outils d'aide à la décision

Quantum electrodynamics:

Some doubts about Feynman's probabilistic presentation

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I. Introduction

We refer here to the book by Richard Feynman "QED: The strange theory of light and matter", 1985. Feynman considers the simple case of a photon going from a point A to a point B, with possible reflection at any place on a plane mirror. We show that the probabilistic axiom introduced by Feynman, namely that the probability of a path is inversely proportional to the square of the length of this path, cannot be correct. The correct presentation must take into account the angles (incident and reflection) and not the length of the path.

II. Simple mathematical concepts

We consider a photon, which goes from a point A to a point B, following a certain path. Following Feynman, we introduce a "probability amplitude", of the form:

$$V = \rho\, e^{i\lambda t}$$

which is a complex number; Feynman represents it as a "spinning arrow".

In this definition:

ρ, modulus of the complex number, is inversely proportional to the length of the path, and may be written under the form:

$$\rho = \sqrt{P}$$

where P is the probability to follow this path.

From this follows obviously that $P = \frac{k}{l^2}$, where k is a constant and l is the length of the path.

In the complex exponential, t is the time needed for the travel from point A to point B and λ is a constant.

We may write:

$$V = \rho \exp\left\{i\, l\lambda / c\right\}$$

where c is the speed of light.

III. A specific situation

Let us consider a specific simple situation, mentioned by Feynman: a photon is emitted in A ; there is a reflection on a plane mirror at any position M and the photon finishes at B, where a detector stands. The problem is supposed to be two dimensional only (the mirror reduces to a line). We want to determine the probability of the path AMB. Let us say that the point A has coordinates $(-1,1)$, point M has coordinates $(x,0)$ and point B has coordinates $(1,1)$. See figure below.

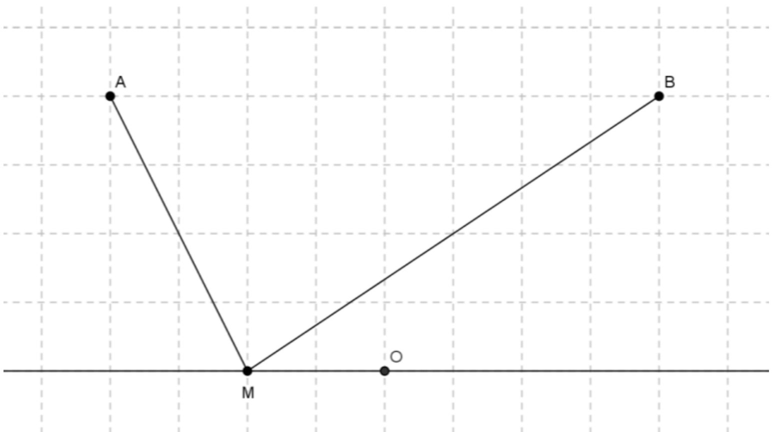


Figure 1: description of a path

The Euclidean length of the path from A to B through M is:

$$l(x) = \sqrt{x^2 + 2x + 2} + \sqrt{x^2 - 2x + 2}$$

The probability $P(x)$ that the photon takes the path through M is therefore:

$$P(x) = \frac{k}{\left(\sqrt{x^2 + 2x + 2} + \sqrt{x^2 - 2x + 2}\right)^2}$$

Our general system of events (also called "universe" in probabilistic terminology) is all possible paths from A to B through any point M. So we have:

$$\int_{-\infty}^{+\infty} P(x)\,dx = 1$$

Let C be defined as:

$$C = \int_{-\infty}^{+\infty} \frac{dx}{\left(\sqrt{x^2+2x+2}+\sqrt{x^2-2x+2}\right)^2}$$

which is a well-defined integral. Numerically, $C \approx 0.599$.

We get:

$$k = \frac{1}{C}$$

and thus:

$$P(x) = \frac{1}{C\left(\sqrt{x^2+2x+2}+\sqrt{x^2-2x+2}\right)^2}$$

We observe (as Feynman did) that this is an even function: the photon has the same probability to take the path through x and the path passing through $-x$. The maximum of probability is obtained for $x=0$.

The probability decreases when x increases, but is never 0; it is possible that the photon takes an extremely long path:

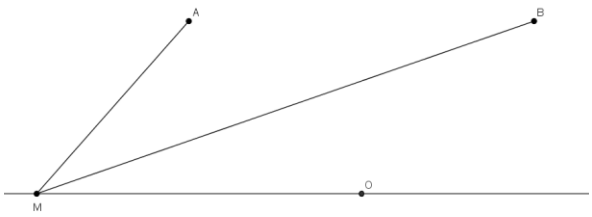


Figure 2: a possible path, with small probability

The probability amplitude vector may be written:

$$V(x) = \sqrt{P}\,\exp\{i\lambda l(x)/c\}$$

where c is the speed of light in the medium and $l(x)$ is the length of the path :

$$l(x) = \sqrt{x^2+2x+2}+\sqrt{x^2-2x+2}$$

We observe that, quite obviously:

$$|V(x)| \sim \frac{1}{2\sqrt{C}\,|x|}$$

which implies that the probability amplitude vectors are not integrable at infinity (only the probabilities are integrable).

IV. Theoretical difficulties

We see here a first theoretical difficulty. We cannot sum up all arrows, but only the arrows corresponding to a probability interval. For instance, we may take a threshold, say 0.95, find the value of α such that:

$$\int_{-\alpha}^{\alpha} P(x)\,dx = 0.95$$

and add up all arrows corresponding to these paths. In other words, we made a truncation to the left and to the right, keeping only the paths with high probability.

Let us compute the resulting arrow, by integrating all paths. We find:

$$V = \int_{-\alpha}^{\alpha} \sqrt{P(x)}\, e^{i\lambda l(x)/c} dx = 2\int_{0}^{\alpha} \sqrt{P(x)}\, e^{i\lambda l(x)/c} dx$$

For the y component of the vector, we obtain:

$$V_y = 2\int_0^{\alpha} \sqrt{P(x)}\, \sin\left(\frac{\lambda l(x)}{c}\right) dx$$

and for the x component:

$$V_x = 2\int_0^{\alpha} \sqrt{P(x)}\, \cos\left(\frac{\lambda l(x)}{c}\right) dx$$

The resulting angle for the arrow is:

$$\vartheta = \arctan\left(\frac{V_y}{V_x}\right) .$$

If we consider simply the path going through the origin, that is $x = 0$, the probability amplitude vector is just:

$$V(0) = \frac{1}{8C}\, e^{\frac{i2\sqrt{2}\lambda}{c}}$$

and the angle of the arrow is:

$$\vartheta_0 = \frac{2\sqrt{2}\lambda}{c}$$

In fact, we see that, playing with different values of α and λ, we may give to ϑ and ϑ_0 any value. There is no reason that ϑ and ϑ_0 should be close to each other.

Let us illustrate this on an example. Take simply $\lambda = c$, and assume that $\sqrt{P}$ is proportional to $1/l(x)$. Then we have:

$$\frac{V_y}{V_x} = \frac{\int_0^\alpha \frac{\sin(l(x))}{l(x)} dx}{\int_0^\alpha \frac{\cos(l(x))}{l(x)} dx}$$

This quotient is not convergent when α increases; in fact, it may take any value. In order to see this, assume that the above limit exists. Then we would have:

$$\lim_{N \to +\infty} \frac{\int_N^{N+1} \frac{\sin(l(x))}{l(x)} dx}{\int_N^{N+1} \frac{\cos(l(x))}{l(x)} dx} = 0$$

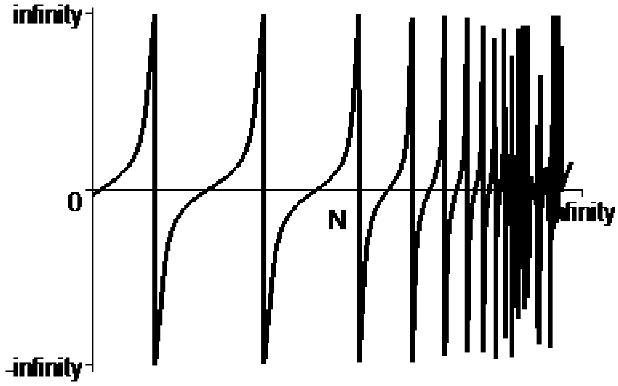
But since $l(x) \sim 2x$ when $x \to +\infty$, this would imply :

$$\lim_{N \to +\infty} \frac{\int_N^{N+1} \frac{\sin(2x)}{x} dx}{\int_N^{N+1} \frac{\cos(2x)}{x} dx} = 0$$

which is not true; the above quantity is asymptotically the same as:

$$\frac{\cos(2N) - \cos(2N+2)}{\sin(2N+2) - \sin(2N)}$$

which oscillates when N tends to infinity, as the following picture shows.



The difficulty here lies in the setting of the axioms taken by Feynman: the assumption according to which the length of the arrow is inversely proportional to the length of the path (page 105, French edition of the book) cannot be correct. Indeed, it leads, as we saw, to a relation of the form $P = \frac{k}{l^2}$. But, in a setting like the one we just saw, the length of the path followed by the photon before the point M and after the point M do not affect the probability; still they affect the length. The photon might continue its trip after B with same probability.

The proper settings are obtained the following way: For any incident angle in Figure 1, the reflection angle may take any value between 0 and π; there is a probability law for the value of the reflection angle, conditioned by the value of the incident angle. The maximum of probability is obtained when the reflection angle is equal to the incident angle.

For any particle, one considers generally that the probability of a path is linked with the number of shocks the particle meets. For instance, for a neutron in a nuclear reactor, a simplified 2d presentation is that the particle may move in any of four directions with probability 1/4:

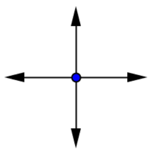


Figure 3: elementary choices of directions

Then, if we consider the path from a point A to a point B, made of N elementary segments, its probability is then $\frac{1}{4^N}$: at each shock, the neutron has four choices and there are N shocks.

The normal situation, in an isotropic medium, is that the probability of a path decreases exponentially with the length of the path. A situation where the decay is inversely proportional to the square of the length can be met only in a non-isotropic medium, as it is the case here. But then the probability depends on the conditions of non-isotropy (here an angle) and not directly on the length of the path.