REAL-TIME TARGET TRACKING :

Optimal determination of the characteristics, using martingale tools

by

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Abstract. – We study the problem of real-time target tracking : how to use all the previous observations in order to get, at a given time, the best knowledge of the position of the target ? A realistic approach is to take into account the processing time for each observation, and if the precision on the position is low, the processing will take longer. As was shown by C. Olivier [1], the tracking is efficient when some affine random walk (characterizing the precision of the tracking) is almost surely bounded. Using martingale techniques, we show that this is the case under proper (and realistic) conditions on the parameters.

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1. Description of the situation

A general situation in real-time target tracking is as follows : the movement of the target x is described by an equation of the form

$$dx = A\,x\,dt + B\,d\omega,$$

where ω is a continuous time random variable, usually assumed to be a vectorial Brownian motion.

If, at time t, an observation Y_t is taken by the camera, this observation itself is not fully deterministic from x_t : it is perturbed by a noise or an imprecision, that is :

$$Y_t = Hx_t + \nu_t \ ,$$

where ν_t itself is also a gaussian variable.

When an observation is made at time t_n, the aim is to use it, as well as all the previous observations, made at earlier times $t_1, \dots, t_{n-1}$, in order to get the best possible knowledge of the position of x at time t_n, that is the conditional expectation

$$E(x_t|Y_{t_1}, \dots, Y_{t_n}) \ , \tag{1}$$

for $t \geq t_n$.

Now, processing the information received from the camera takes a certain time, and one cannot start treating the $n+1$-st observation before the treatment of the n-th is finished. In order to reduce this time, one considers only a smaller window – a rectangle in which the target is most likely to be. This leads to the introduction of a new parameter, a probability p : the probability that the target is indeed in this smaller window. If this is the case, the processing will be faster. In the other case, with probability $1-p$, we have either to enlarge the window or to look at other windows, and both take more time.

The processing time of the nth observation depends on one more parameter, namely the accuracy selected for the analysis of the image, or, in other words, the pixel size. A low accuracy means an aggregation of pixels into larger homogeneous zones, which are considered as elementary during the analysis phase : this reduces the processing time. The precise tuning of this parameter will be considered below.

The tracking procedure will be considered as satisfactory if the uncertainty (the covariance matrix) of the target state estimator remains bounded when time elapses.

In the mono-dimensional case (state and observer being scalars), both evolution and observer equations reduce to

$$dx = \lambda\,x\,dt + d\omega,$$

$$Y_t = x_t + \nu_t \ ,$$

where λ is a real number and ν_t is an observation error, assumed to be gaussian and centered. The variance of ν_t is the selected resolution parameter at time t, denoted by ρ_t^2. The condition "the uncertainty should remain bounded" thus translates into a requirement upon the variance X_k of x at time k, namely that there exists a constant M, such that

$$X_k \leq M, \tag{2}$$

for all k. Since a small number of observations at the beginning do not really matter, it is enough to have (2) only for k greater than some k_0.

We will consider only the case $\lambda \geq 0$, which corresponds to naturally unstable processes. The other case ($\lambda < 0$) is completely straightforward, and the variance is always bounded.

We now refer to C. Olivier [1] for a complete description of the equations. Let us simply say here that, under proper assumptions, the variance X_k is governed by the relations (respectively a Lyapunov and a Riccati equation) :

$$
\begin{aligned}
X_k &= e^{2\lambda\tau_{k-1}} \frac{X_{k-1}\rho_{k-1}^2}{X_{k-1}+\rho_{k-1}^2} + \frac{\Omega^2}{2\lambda}(e^{2\lambda\tau_{k-1}}-1), \qquad \text{with probability } p,\\
X_k &= e^{2\lambda\tau_{k-1}} X_{k-1} + \frac{\Omega^2}{2\lambda}(e^{2\lambda\tau_{k-1}}-1), \qquad \text{with probability } 1-p,
\end{aligned}
$$

where τ_{k-1} is the processing time at time $k-1$ (it depends on the variance X_{k-1} and on the resolution ρ_{k-1}) ; Ω is a Brownian motion and λ is a positive real constant : both come from the equations (1) of the target motion.

If we decide that the resolution will always be proportional to the precision, that is

$$\rho_t^2 = \beta X_t , \qquad \text{for all } t,$$

and if we perform the sampling at constant intervals, the above equations become linear and reduce to :

$$X_k = \begin{cases} a_1 X_{k-1} + b_1 & \text{with probability} \quad p \\ a_2 X_{k-1} + b_2 & \text{with probability} \quad 1-p \end{cases} \tag{3}$$

where a_1, a_2 are positive real numbers such that $0 < a_1 < 1 < a_2$, and b_1, b_2 are positive real numbers. The problem is now to show that if the parameters a_1, a_2, b_1, b_2 are correctly “tuned”, the tracking precision will not eventually deteriorate.

2. The probabilistic description of the problem

We are looking for sufficient conditions ensuring almost sure boundedness for a non-negative stochastic process X_k, governed by the “affine random walk” equations (3). At time $k = 0$, the process is at X_0.

For $M > 0$, we define the stopping time T_M as the first k for which $X_k \le M$ (and $T_M = +\infty$ if $X_k > M$ for all k).

We are going to prove :

Proposition 1. – *Assume*

$$p \log a_1 + (1-p) \log a_2 < 0. \tag{4}$$

Set $\Delta = \dfrac{b_2 - b_1}{a_2 - a_1}$, *and*

$$b = b_1 + \Delta(1-a_1) = b_2 + \Delta(1-a_2).$$

If M is chosen large enough, namely

$$M > \frac{b}{1 - a_1^p a_2^{1-p}} - \Delta, \tag{5}$$

then, for any starting point $X_0 > M$, we have

$$ET_M \le \frac{\log(X_0+\Delta) - \log(M+\Delta) - \log a_1}{\log \frac{1}{a_1^p a_2^{1-p}} - \log \frac{M+\Delta}{M+\Delta-b}} .$$

This implies that, no matter where the stochastic process starts, almost surely it will enter the strip $0 < y < M$. If we start at $X_0 \le M$, then $T_M = 0$, but the result implies that, if X_k leaves the strip, it will eventually come back to it, with probability 1.

Remark. – We observe that condition (4) is weaker than

$$pa_1 + (1-p)a_2 \ < \ 1,$$

which is itself weaker than $p + (1-p)a_2 < 1$. This latter condition would be given by considerations from Liapunov exponents, but it cannot be satisfied in practice, since the coefficient a_2 has to be strictly bigger than 1.

We now turn to the proof.

Proof. – First, we make some simple reductions. With $\Delta = \dfrac{b_2 - b_1}{a_2 - a_1}$, define X'_k by $X'_k = X_k + \Delta$. Then it satifies

$$X'_k \ = \ \begin{cases} a_1 X'_{k-1} + b & \text{with probability} \quad p \\ a_2 X'_{k-1} + b & \text{with probability} \quad 1-p \end{cases}$$

where

$$b \ = \ b_1 + \Delta(1-a_1) \ = \ b_2 + \Delta(1-a_2) \ = \ b_1 \frac{a_2 - 1}{a_2 - a_1} + b_2 \frac{1 - a_1}{a_2 - a_1}.$$

The conditions $b_1 > 0$, $b_2 > 0$, $a_1 < 1 < a_2$ ensure that $b > 0$. So next we consider

$$X''_k \ = \ \frac{1}{b} X'_k \ ,$$

which satisfies

$$X''_k \ = \ \begin{cases} a_1 X''_{k-1} + 1 & \text{with probability} \quad p \\ a_2 X''_{k-1} + 1 & \text{with probability} \quad 1-p \end{cases}$$

Moreover,the event $\{X_0 > M, \ldots, X_k > M\}$ is just $\{X''_0 > (M+\Delta)/b, \ldots, X''_k > (M+\Delta)/b\}$. So we need only consider the case $b_1 = b_2 = 1$, for which we return to our original notation X_k. This means that we consider a process X_k on $\mathbb{R}_+$, defined by

$$X_{k+1} \ = \ R X_k + 1, \tag{6}$$

where R takes the values a_1 and a_2 with respective probabilities p and $1-p$.

Definition. – The process X_k being defined as above, the process Y_k is said to be linked to X_k if

$$Y_{k+1} \ = \ R Y_k, \qquad Y_0 = X_0 \ , \tag{7}$$

We see that the link between (X_k) and (Y_k) is fully deterministic : if we happen to know a trajectory $X_0, X_1, \ldots, X_k$, we know the trajectory $Y_0, Y_1, \ldots, Y_k$, and conversely. This will enable us to convert a problem dealing with the first into a problem dealing with the second, which is much easier since $\log Y_k$ is just a random walk.

Let r_n be a sequence of i.i.d. random variables taking values a_1 and a_2 with respective probabilities p and $1-p$: r_n is just the n-th realization of R.

Using the r_n's, we can write the two processes :

Lemma 2. – *For all* $n \geq 1$,

$$X_n \ = \ 1 + r_n + r_n r_{n-1} + \cdots + r_n r_{n-1} \cdots r_1 X_0 \tag{8}$$

$$Y_n \ = \ r_n r_{n-1} \cdots r_1 X_0 \tag{9}$$

Proof : this is easy by induction.

We can now describe the links between the two processes :

Lemma 3. – *The processes* (X_n) *and* (Y_n) *are related through the following equations :*

$$Y_n = (X_n - 1)\frac{X_{n-1}-1}{X_{n-1}} \cdots \frac{X_1 - 1}{X_1}\ , \qquad n \geq 2 \tag{10}$$

$$X_n = Y_n(1 + 1/Y_1 + \cdots + 1/Y_n)\ , \qquad n \geq 1 \tag{11}$$

Proof of lemma 3. – Since $X_n = r_n X_{n-1} + 1$, we get $r_n = (X_n - 1)/X_{n-1}$; replacing in (9) yields (10) (note that $X_n > 1$ for all $n \geq 1$). The same way, $r_n = Y_n/Y_{n-1}$ and thus

$$\begin{aligned} X_n &= 1 + \frac{Y_n}{Y_{n-1}} + \frac{Y_n}{Y_{n-1}}\frac{Y_{n-1}}{Y_{n-2}} + \cdots + \frac{Y_n}{Y_{n-1}}\frac{Y_{n-1}}{Y_{n-2}} \cdots \frac{Y_1}{Y_0} Y_0 \\ &= 1 + \frac{Y_n}{Y_{n-1}} + \frac{Y_n}{Y_{n-2}} + \cdots + \frac{Y_n}{Y_1} + Y_n, \end{aligned}$$

which is (11).

The next step is to a find an upper bound for the probabilities of events of the form $(X_0 > x, X_1 > x, \ldots X_k > x)$, by means of the associated random walk $\log Y_k$. First, we have

Lemma 4. – *The set of conditions* $X_0 > x$, …, $X_n > x$ *imply* $Y_0 > y_0$, …, $Y_n > y_n$, *with*

$$y_k = (x-1)\,(1 - 1/x)^{k-1}\ , \qquad k \geq 1.$$

Proof of lemma 4. – With $y = (x-1)/x = 1 - 1/x$, which is an increasing function of $x > 0$, we have, if $X_k > x$, by (10) :

$$Y_k > (x-1)\left(\frac{x-1}{x}\right)^{k-1} = (x-1)\left(1 - \frac{1}{x}\right)^{k-1}\ , \qquad k \geq 1.$$

This proves the lemma.

We denote by $M_k = \log Y_k$ the random walk associated to Y_k. We have

$$M_k = \begin{cases} M_{k-1} + \log a_1 & \text{with probability} \quad p \\ M_{k-1} + \log a_2 & \text{with probability} \quad 1-p \end{cases} \tag{12}$$

and, by Lemma 4 :

Corollary 5. – *If* $X_0 > M$, …, $X_n > M$, *then for all* $k = 0, \ldots, n$,

$$M_k > \log(M-1) + (k-1)\log(1 - 1/M).$$

This can be rewritten

$$M_k > \log M - k \log \frac{M}{M-1}. \tag{13}$$

Let τ be defined by

$$\tau = \inf\{n\ ;\ M_n \leq \log(M-1) + (n-1)\log(1 - \frac{1}{M})\}.$$

Then, by Corollary 5 :

$$\begin{aligned}\{T_M > n\} &= \bigcap_{k=0}^{n}\{X_k > M\} \\ &\subset \bigcap_{k=0}^{n}\{M_k > \log(M-1) + (k-1)\log(1-1/M)\} \\ &= \{\tau > n\},\end{aligned}$$

and thus

$$ET_M = \sum_0^\infty P\{T_M > n\} \le \sum_0^\infty P\{\tau > n\} = E\tau. \tag{14}$$

Now, with $\lambda = -(p\log a_1 + (1-p)\log a_2)$, we have, by (12),

$$E(M_{n+1}|M_n) = M_n - \lambda. \tag{15}$$

Let $Z_n = M_n + n\lambda$. It follows from (15) that Z_n is a martingale, with respect to the natural filtration. So the stopped process $(Z_{k\wedge\tau})_{k\ge 0}$ is also a martingale, and

$$EZ_{k\wedge\tau} = Z_0 = \log X_0 , \tag{16}$$

that is

$$EM_{k\wedge\tau} + \lambda E(k\wedge\tau) = \log X_0 . \tag{17}$$

But by (13),

$$M_{k\wedge\tau} > \log M + \log a_1 - (k\wedge\tau) \log\frac{M}{M-1}.$$

Indeed, at time τ, when M_k steps the first time below M, the descent is at most $\log a_1$. This inequality implies

$$EM_{k\wedge\tau} > \log M + \log a_1 - \log\frac{M}{M-1} E(k\wedge\tau). \tag{18}$$

Comparing with (17), we get

$$\log X_0 - \log M - \log a_1 > (\lambda - \log\frac{M}{M-1}) E(k\wedge\tau).$$

Since $\log X_0 > \log M$, we need to take M in such a way that $\lambda > \log\frac{M}{M-1}$, which requires $\lambda > 0$ (guaranteed by (4)) and means

$$M > \frac{1}{1 - a_1^p a_2^{1-p}} . \tag{19}$$

If M is chosen this way, we get

$$E(k\wedge\tau) < \frac{\log X_0 - \log M - \log a_1}{\log\frac{1}{a_1^p a_2^{1-p}} - \log\frac{M}{M-1}} ,$$

and, by (14),

$$\begin{aligned}ET_M &\le E\tau \\ &\le \liminf_{k\to\infty} E(k\wedge\tau) \\ &\le \frac{\log X_0 - \log M - \log a_1}{\log\frac{1}{a_1^p a_2^{1-p}} - \log\frac{M}{M-1}} ,\end{aligned}$$

which proves our result.

Conclusion

We derived an "affine" random walk formulation from a concrete target tracking problem and showed, using martingale tools, that the associated process is almost surely bounded.

Reference

[1] C. Olivier : Real-time observability of targets with constrained processing power. *To appear.*