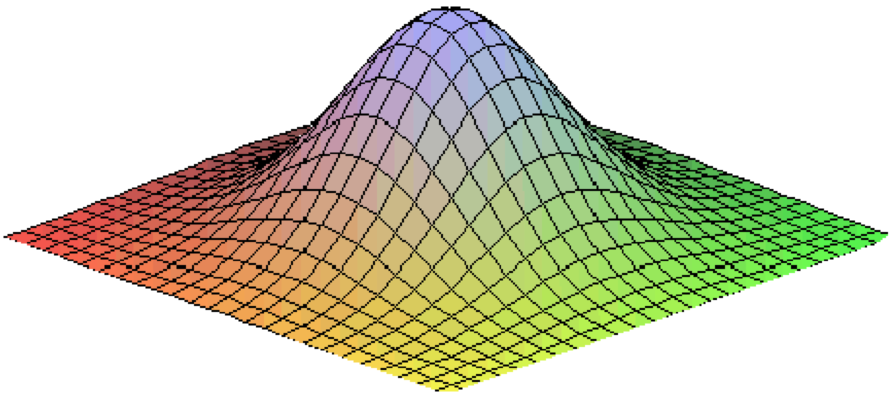
Simple Random Walks

In the Plane:

An Energy-Based Approach

by Bernard Beauzamy

Société de Calcul Mathématique SA
(Mathematical Modelling Company, Corp.)



Real Life Mathematics

Société de Calcul Mathématique SA
(Mathematical Modelling Company, Corp.)

111 Faubourg Saint Honoré, 75008 Paris France

tel +33 1 42 89 10 89, fax + 33 1 42 89 10 69.
email: contact@scmsa.com

www.scmsa.eu

Simple Random Walks

In the Plane:

An Energy-Based Approach

by Bernard Beauzamy

Collection "Real Life Mathematics"

Société de Calcul Mathématique SA
(Mathematical Modelling Company, Corp.)
Tools for Decision Help
since 1995

By the same author:

Espaces d'Interpolation réels : Topologie et Géométrie
Springer-Verlag, Lecture Notes no 666, 1978.

Introduction to Banach Spaces and their Geometry
North Holland, Collection « Notas de Matematica », vol. 68.
Première édition : 1982, seconde édition : 1985.

Modèles Etalés des Espaces de Banach
(en collaboration avec J.T. Lapresté)
Editions Hermann, collection « Travaux en Cours », 1984.

Introduction to Operator Theory and Invariant Subspaces
North Holland, Mathematics Library, 1988.

Méthodes probabilistes pour l'étude des phénomènes réels
ISBN 2-9521458-0-6, édité par la SCM, 2004. Second Edition, 2014.

Comment décider et gérer un programme de recherche scientifique? Manuel pratique à l'usage des entreprises
ISBN: 2-9521458-1-4, édité par la SCM, 2005.

Méthodes probabilistes pour la reconstruction de données manquantes
(en collaboration avec Olga Zeydina)
ISBN: 2-9521458-2-2, ISSN 1767-1175, édité par la SCM, 2007.

Nouvelles méthodes probabilistes pour l'évaluation des risques
ISBN 978-2-9521458-4-8. ISSN 1767-1175, édité par la SCM, 2010.

Archimedes' Modern Works
ISBN 978-2-9521458-7-9. ISSN 1767-1175, édité par la SCM, 2012.

Probabilistic Information Transfer
(en collaboration avec Olga Zeydina)
ISBN 978-2-9521458-6-2, ISSN 1767-1175, édité par la SCM, 2013.

Méthodes Probabilistes pour la Gestion des Risques Extrêmes
ISBN: 978-2-9521458-9-3 ISSN: 1767-1175, édité par la SCM, 2015.

*****

ISBN: 979-10-95773-01-6 **ISSN:** 1767-1175

*****

Ouvrage édité par la Société de Calcul Mathématique SA
Siège social et bureaux: 111, Faubourg Saint Honoré, 75008 Paris, France
Tel.: +33 1 42 89 10 89. Fax: +33 1 42 89 10 69. email: contact@scmsa.com
Société Anonyme au capital de 46 200 Euros. RCS : Paris B 399 991 041.
SIRET: 399 991 041 00035. APE: 731Z.

Simple Random Walks

In the Plane:

An Energy-Based Approach

Introduction

I. Strange from the very beginning

Everyone knows how to flip a coin and so everyone understands what a game of heads and tails is. It can be linked with money: each player chooses one side and, if this side comes, he receives one Euro from the other player. This is a simple game, quite honest (both sides of the coin have the same probability) and it may be played everywhere on Earth, and probably also on any planet with gravity.

Quite clearly, the probability to win, for each player, is the same, namely 1/2. So, apparently, there is nothing to discuss. But things become more interesting if we investigate repetitions of the game (what we call "iterations", in mathematical terms). Most people, in this case, think that, since the game is well-balanced, the gain of each player should tend to zero. It turns out that this is not true at all.

In practice, moreover, each player has an initial fortune, which is of course finite. One of the players may start with, let us say, 100 Euros and the other with 200 Euros. So, each of them may wonder: how long will the game last, before I get ruined or before my partner does? Is this time proportional to the initial fortunes? Say, I start with 100 Euros and I decide to play during 200 games. What "fortune" may I expect at the end? Would it be different if I started with 150 Euros?

Mathematically speaking, the game of heads and tails may be described as a "simple random walk" in the plane: a sequence of independent random variables X_n with values ± 1, with probability $1/2$ in each case. The variable X_n describes the result of the n^{th} game.

The result of the sequence $\left(X_1,\dots,X_N\right)$ is easy to describe: it may be any sequence of ± 1, and each such sequence has probability $1/2^N$. So, the sequence in itself is not our concern; we are interested in the sum, representing the gain.

Let $S_N = \sum_{n=1}^{N} X_n$ be the sum of the first N variables. The sum S_N represents the increase of fortune of A compared to B at the end of N games; this increase may of course be positive or negative. At the initial moment, we set $S_0 = 0$. At any time N, the values of S_N are necessarily between $+N$ and $-N$. The first situation occurs when the first player wins all the time; the second situation when he loses all the time.

Of course, since the variables take the values ± 1 with same probability, the mathematical expectation satisfies $E\left(S_N\right) = 0$ and since they are independent, the variance satisfies $\mathrm{var}\left(S_N\right) = N$.

Here is a possible output, for 100 games; what is drawn is the algebraic gain of the player A from the beginning:

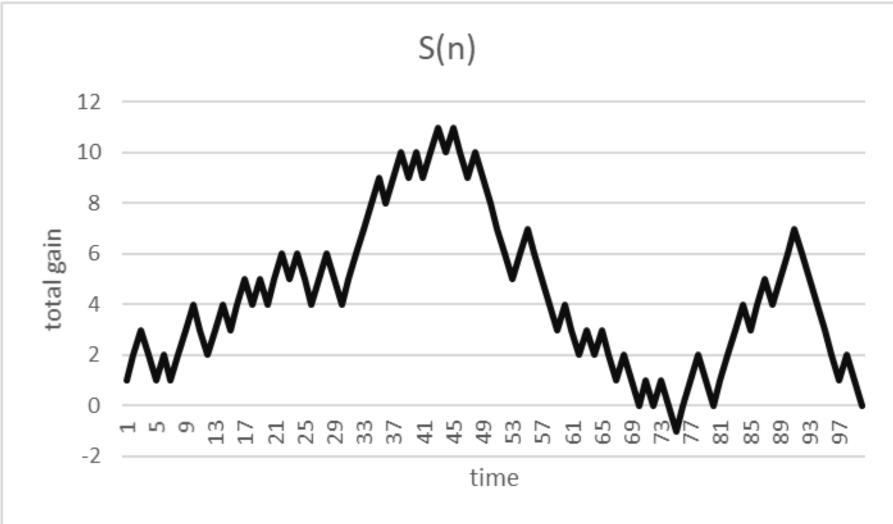


Figure 1: The gain of the first player

Let us do it again from the very beginning: run again a set of 100 independent games. We see that the output may be completely different, though the rules were exactly identical:

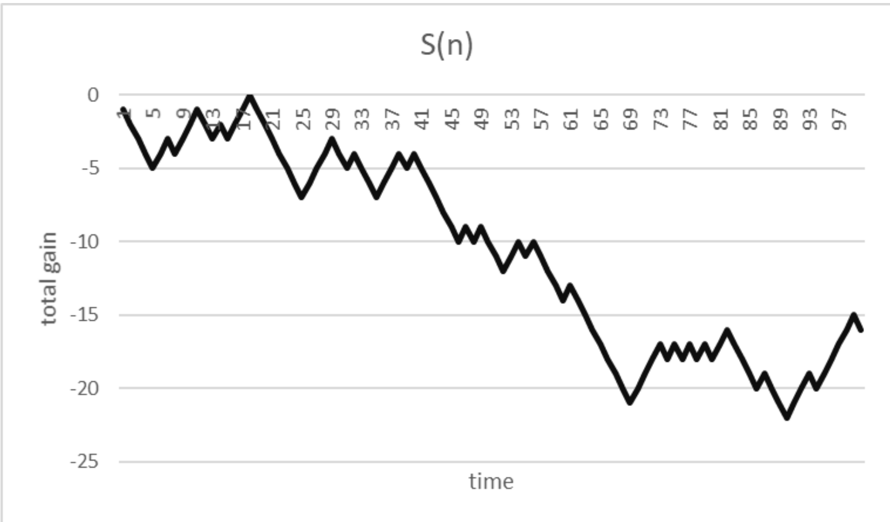


Figure 2: The gain of the first player, second version

This is very interesting: we see that a probabilistic experiment does not satisfy the "repeatability" conditions which are usually required for any experiment in physics. Indeed, researchers usually consider as a basic law for scientific research that any experiment should be "repeatable", namely if you proceed to the same experiment twice, under the same conditions, with same data, it should provide the same result. Well, this is not the case for any experiment which is linked with probabilities. And since probabilities appear fundamentally in most laws of Nature (starting with Quantum Physics), one may believe that the "repeatability" requirement is not valid.

Let us come back to our game.

Independently of the game, each player has an initial fortune, which is finite, or infinite in a theoretical setting. The game may stop when one of the players is ruined (his fortune becomes equal to 0). The general question is to study the behavior of S_N (possible values, with their probabilities), the duration of the game, depending upon the initial fortunes, and the asymptotic behavior, when $N \to +\infty$.

For instance, many people think that $S_N \to 0$ when $N \to +\infty$, but this is completely wrong. There is often a confusion with the fact that, according to the law of large numbers, the average $\frac{S_N}{N} \to 0$ when $N \to +\infty$. It turns out, in fact, that the sum S_N tends to "oscillate", with oscillations which are larger and larger. This very strange fact will be made clear in the future.

The behavior of S_N is determined by laws of Nature: one may repeat the experiment and check the results. But, at the same time, these laws are axiomatically defined, as we just did. Such random walks are probably the only example of laws of Nature which may be axiomatically defined: all laws in Physics are otherwise empirical. This remark, in itself, justifies a careful study of the situation.

Also, the strange fact that the sums tend to oscillate is interesting from a philosophical point of view. Many people tend to think that Nature eventually goes to an equilibrium (in terms of energy, of repartition of species, economic power, and so on); the results about Random Walks tend to show that, on the contrary, Nature proceeds by large variations, with increasing amplitude. This is fundamentally different.

So, we claim that the analysis of Simple Random Walks in the plane, which, at first sight, looks very technical, carries in fact two very important conclusions:

- The "repeatability" requirement is incorrect;
- Nature does not look for an equilibrium but proceeds by large oscillations.

II. Existing results

The best known one is Khintchin's law of the iterated logarithm (1924). Let us define a "Khintchin's curve" by the equation $\varphi(x)=\sqrt{2xLog\left(Log(x)\right)}$; then the Law of the Iterated Logarithm deals with the asymptotic behavior of the quotient $\dfrac{S_N}{\varphi(N)}$ when $N \to +\infty$, and says that the $\limsup$ of this quotient is $+1$ and the $\liminf$ is -1, when $N \to +\infty$. This, indeed, already ensures that S_N "oscillates", because it must come back infinitely many times close to $\varphi(x)$ and close to $-\varphi(x)$. But, as they are stated, such results are probabilistic in nature and not quantitative at all. Detailed statements and proofs will be given later in this book.

The probabilistic appearance of Khinchin's laws is misleading. Looking at such a statement, everyone has the feeling that, for a given player, there are some unknown forces which will, sooner or later, bring his fortune close to Khintchin's curve and to its opposite. This is completely wrong; at any time, the game is only governed by the ± 1 rule, with equal probability.

What Khintchin's laws say, and, more generally, what any result about random walks says, is that there are more paths with some properties than paths with other properties. They are not individual results about each path; they are results about the number of paths with a given characteristic. Such results are in fact of combinatorial nature.

III. Our approach

We present here a new approach to such problems, which is "energy based" and not probabilistic. Instead of counting paths, we consider that there is a propagation of energy, and the more energy is received by a point, the more paths lead to this point.

We introduce the notion of a "barrier", which is not new (see [Feller]): this is a curve (for instance an horizontal line $y = y_0$) which has the property to be "energy absorbing": if the sum S_N reaches this barrier, it is annihilated; the energy disappears. Physically speaking, it corresponds to the ruin of one of the players. For instance, if the barrier has been set to $y=1000$, and if S_N reaches it, it means that the first player has won 1000 Euros, so the second player has lost 1000 Euros, and if the initial fortune of the second player was precisely this amount, the second player is ruined and the game stops. So, the concept of "barrier" is well-adapted to describe the ruin of one of the players.

Of course, there may be an upper (for $y>0$) and a lower barrier (for $y<0$), describing the fact that both players may not have the same initial fortune.

Also, the barrier may vary with time (for instance $b(x)=\sqrt{x}$), which means physically that the players do not have a constant fortune; this fortune increases with time. This allows us to see the connection with Khintchin's results.

This analytic framework will allow us to develop a unified framework, and to obtain quantitative estimates which were not known previously. We turn a probabilistic problem into an analytic one, for which we may use classical tools from analysis: Operator Theory, eigenvalues, eigenvectors, Chebycheff polynomials of first and second kind. Most computations turn out to be of trigonometric nature, which is rather unexpected in this context, but the final results are rather simple.

IV. Organization of the book

The book is organized in five Parts:

Part I: Basic Concepts
Part II: Identical Initial Fortunes
Part III: Different Initial Fortunes
Part IV: Variable Fortunes
Part V: Khinchin's Law of the Iterated Logarithm: Quantitative versions