Société de Calcul Mathématique SA
Outils d'aide à la décision

Fédération Française des Jeux Mathématiques

Mathematical Competitive Game 2014-2015

Uncertainties in GPS Positioning

Fédération Française des Jeux Mathématiques
(French Federation of Mathematical Games)

and

Société de Calcul Mathématique SA

in partnership with

The French Institute for Transportation Science and Technology, Geolocalisation Team (IFSTTAR/CoSys/Geoloc)

and

The French Ministry of Transportation, Mission for Tarification Pricing (MEDDTL/DIGITIM/SAGS/MT)

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I. Presentation of the Games

The "Mathematical Games", jointly organized by FFJM and SCM, have existed for six years; the previous ones were:

- In 2008-2009, conception of a bus transportation network in a city, in partnership with Veolia Transport;
- In 2009-2010, conception of an electricity distribution network, in partnership with RTE (Réseau de Transport d'Electricité);
- In 2011-2012, search for the best itinerary by a car, in partnership with the newspaper Auto Plus;
- In 2012-2013, fighting forest fires in Siberia, in partnership with the Paris Firemen Brigade;
- In 2013-2014, checking an industrial process.

They deal with the resolution of a "real life" problem, that is a problem of general concern, but simplified in its mathematical contents. Still, the resolution typically requires several months of work.

Candidates may compete individually or as groups, for instance high school classes, or college students, or university students, preparing a "memoir" for the end of their studies.

Two categories of prizes are given:

Individual prizes:
For the winner: 500 Euros
For the second: 200 Euros
For the next three: 100 Euros each.

Prizes for groups:
For the winner: 500 Euros
For the second: 200 Euros
For the next three: 100 Euros each.

The total amount of prizes is therefore 2 000 Euros. The best solutions are published on the web site of FFJM, on the web site of SCM, and on the web sites of our partners. The official announcement of the results and the ceremony for prizes occur during the "Salon de la Culture et des Jeux Mathématiques" (Fair for Mathematical Culture and Games), which is held in Paris, each year, during the month of May.

The winners, previous years, gained considerable notoriety, both in the press and television in their respective countries.

II. The 2014-2015 Prize

A. General presentation of the subject

Everyone, nowadays, knows what a GPS receiver is: it receives a signal from several satellites, and, using this information and a built-in map, it tells you where you are on the map. The computation is usually quite fast.

The question which is seldom addressed is: what is the uncertainty on this position, and how is it computed ? This is the topic of the present Game.

Such uncertainties may have important consequences. Let us recall that, in 1682, a geographer named Philippe de La Hire published a new map of France, with the title "Carte de France corrigée par ordre du Roi sur les observations de Mrs de l'Académie des sciences" (map of France, corrected under an order of the King, following the observations of the Academy of Sciences). France was narrower on this map, so the king Louis XIV said : "The Academy of Sciences deprived France of more territory than all its enemies combined ever did !".

B. Satellites

In what follows, we consider that 5 satellites are in operation and can transmit to the receiver. Everything is considered as static: the satellites, the receiver, the Earth. We consider a problem which is purely instantaneous (which means, for instance, that the knowledge of a previous position will not be used in order to improve the knowledge of a future position).

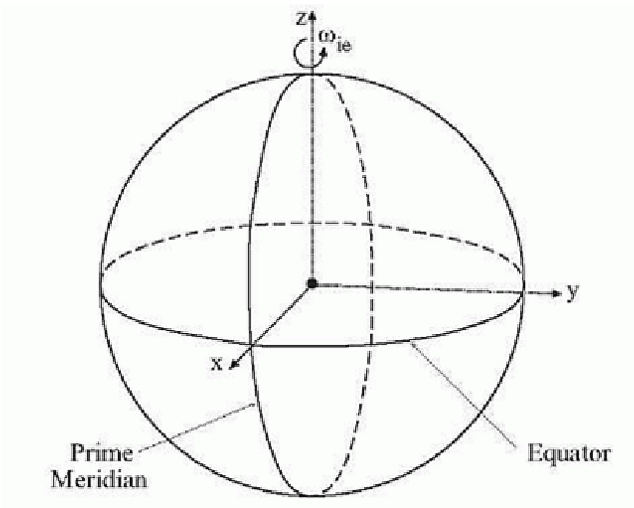

Figure 1 : the axes connected with the Earth

The reference axes will be drawn as on the figure above. The Earth is considered as a ball with center Ω ; the x axis goes through the equator and the Greenwich meridian ; the y axis goes through the equator and is orthogonal to the x axis after a rotation of $+\frac{\pi}{2}$ and the z axis goes through the North Pole. Such a choice is completely standard.

C. Signals sent by the satellites

We have 5 satellites, and each satellite S_k, $k=1,\ldots,5,$ sends a signal, which is analyzed by the receiver R ; the clocks of the satellites are atomic clocks, supposed to be perfect and synchronized (this is the "absolute time") ; the clock of the receiver is an ordinary clock, which may differ from the absolute time by a value τ which may be positive or negative, and is unknown.

Let $d_k = dist(R, S_k)$ be the (unknown) distance from the $k-$th satellite to the receiver. Let c be the speed of light ($c \approx 3\times 10^8$ m/s). From the signal sent by each satellite, the receiver thus deduces the time $\frac{d_k}{c}+\tau.$ In other words, we know the quantities $\delta_k = d_k + c\tau,$ for $k=1,\ldots,5.$ These quantities are called "pseudo-distances" in what follows.

In order to compute these pseudo-distances, there are many corrections (effect of relativity, effect of the atmospherical conditions, and so on), which are neglected here.

D. Solving the usual problem

The unknown coordinates of the receiver are denoted as (x_0, y_0, z_0) and the coordinates of the $k-$th satellite, known up to some error, are denoted as (x_k, y_k, z_k).

The receiver has 5 equations of the form :

$$\left((x_0 - x_k)^2 + (y_0 - y_k)^2 + (z_0 - z_k)^2\right)^{1/2} + c\tau = \delta_k \qquad (1)$$

From these 5 equations, the receiver computes the 4 unknowns : $x_0, y_0, z_0, \tau.$ The way to solve this system of equations is standard. One possibility is to take differences of two such equations, which gives a set of equations in which τ has disappeared.

E. Difficulties

In practice, the position S_k of each satellite is not perfectly known, neither is the pseudo distance δ_k to the receiver. A probabilistic approach will be used in order to attribute a position to each satellite and a value to each pseudo-distance. At the end, this probabilistic approach will allow to obtain a 3 dimensional probability law for the receiver, and to define a volume where it has 90% chances to be.

F. Rough data

The rough data are as follows:

	coordinates in meters			
	X	Y	Z	Pseudo-distance satellite - receiver (in meters)
satellite 1	15 470 963.50	-1 180 726.85	21 541 839.41	20 260 438.90
satellite 2	19 603 002.46	4 726 671.62	17 059 949.34	20 264 387.99
satellite 3	3 017 916.51	15 760 014.60	21 367 885.75	23 104 936.52
satellite 4	180 842.82	-15 551 720.30	21 714 117.29	23 382 913.05
satellite 5	25 616 941.53	7 756 572.54	336 879.28	23 101 783.81

Table 2 : the rough data

In the table above, the columns 2,3,4 are the coordinates of the point S_k and the 5-th column is the value of the pseudo-distance δ_k. All numbers are in meters.

G. Probabilistic laws

1. Upon the position of each satellite (columns 2,3,4, table 2)

We will consider that the uncertainty is limited to a sphere of radius 2 meters around the points indicated in table 2 above. In other terms, each satellite must be at a point S'_k satisfying $dist(S'_k, S'_k) \leq 2$, for $k = 1,...,5$. Here, the distance is simply the euclidean distance :

$$dist(X,Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

if $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$.

But the probability law is not uniform. It will be distributed as follows:

distance to the center point (in meters)	probability (percentage)
0-0.40	30%
0.40-0.80	25%
0.80-1.20	20%
1.20-1.60	15%
1.60-2	10%

Table 3 : distribution of probability according to the distance

In other words, each satellite has 3 times more chances to be near the announced position (error less than 0.40 m) than to be far away (error more than 1.60 m).

2. Upon the pseudo-distance between the satellite and the receiver (column 5, table 2)

We consider that the pseudo-distance may be at most ten meters lower than announced, or ten meters bigger. In other words, the absolute value of the error cannot exceed ten meters.

But again the probabilities are not uniform in this range ; they are evaluated as follows, and will be identical for all satellites :

pseudo-distance to the receiver (in meters)	probability
d-10 to d-8	5%
d-8 to d-6	7.5%
d-6 to d-4	10%
d-4 to d-2	12.5%
d-2 to d	15%
d to d+2	15%
d+2 to d+4	12.5%
d+4 to d+6	10%
d+6 to d+8	7.5%
d+8 to d+10	5%

Table 4 : distribution of probability for the distance

In this table, d is the distance announced by the receiver, that is the 5-th column of table 2. The distance is different for all 5 satellites, but the repartition of the errors is the same for all.

In each sub-interval, the probability distribution will be considered as uniform. For instance, we have no reason to think that the pseudo-distance may be more likely between d+4 and d+5, rather than between d+5 and d+6. We treat the whole interval d+4 to d+6 in a global manner, and the same for all intervals.

H. Independence

We will assume (though this is not completely correct) that the error on the position of the satellite is independent from the error on the distance, and that all satellites are mutually independent. This is not completely correct, because all these computations are performed by the same receiver, which might have some defect, affecting all its computations.

III. Questions

- What is the expected position of the receiver ?
- Describe a set where the receiver has 90% chances to be.

IV. Participation rules

The game starts on November 1st, 2014 and ends on April 30th, 2015. Prizes will be given in May 2015, during the "Salon des Jeux Mathématiques", in Paris.

Participants should send their solution, in pdf format, in English or in French, no later than April 30th, 2015, to the email address: **ffjm@wanadoo.fr.**

No preliminary registration is required. Everyone can participate.