**Eigenvalue Assignment Connected with Output Feedback**

by

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**Abstract**. – We show that the well-known problem of the assignment of the spectrum $\sigma(A+BKC)$, where $A\ (n\times n)$, $B\ (n\times m)$, $C\ (p\times n)$ are fixed matrices, $K\ (m\times p)$ arbitrary, all with real entries, can be solved in general if $\min(m,p)\geq c\sqrt{n}$. The numerical constant $c$, depending on various parities and on the nature of the set to be assigned, takes values between $2\sqrt{2}$ and $4\sqrt{3}$. Previously known result (Kimura, 1975) was $m+p>n$.

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**Introduction**.

Let $A\ \ (n\times n)$, $B\ \ (n\times m)$, $C\ \ (p\times n)$ be three fixed matrices, with real entries. We assume that $m\leq n$, $p\leq n$, and that $B$ and $C$ have full rank.

Let $K\ \ (m\times p)$ be an arbitrary matrix, with real entries, at our disposal. The eigenvalue-assignment problem, connected with output feedback, is : playing with the $m\times p$ entries in $K$, where can we put the spectrum of the matrix $A+BKC$ ? More precisely, given any set $S$ of complex numbers $\{\lambda_1,\dots,\lambda_n\}$ such that the set of conjugates $\overline{S}$ is equal to $S$, can we find $K$ such that

$$\sigma(A+BKC)=S \quad ?$$

This problem comes from Control Theory and, though it received considerable attention (see for instance J-F Magni [4] and Champetier-Magni [2], and the references in these papers), only limited progresses have been made. In [2] was proved that if $m+p>n$, the answer to the problem is "yes", except for exceptional situations of the poles.

Here we show that the entire spectrum assignment problem can be solved in general (that is : except for exceptional positions of the $\lambda_j$'s) as soon as $\min(m,p)\geq\sqrt{8n+33}-4$ (when the set to be assigned is real for more than one half), or $\min(m,p)\geq 2\ \sqrt{12n+181}-25$ (in the other case). Minor variations due to parity of $n$ and $\min(m,p)$ appear ; the precise statements are given in theorems 3.1 and 4.1 below.

The paper is divided into two parts, which are of fairly different nature. In the first part, we show that the assignment problem (which is of operator-theoretic nature) can be brought back, in an equivalent manner (that is, with no loss of information) to a problem of intersection of linear subspaces : given subspaces $F_1,\dots,F_n$ of $\mathbb{R}^n$ or $\mathbb{C}^n$, with prescribed positions and dimensions, can we find one linear subspace $H$, with given dimension, that intersects them all ? (by "intersects", we mean of course non-trivially : $H\cap F_j\neq\{0\}$, for all $j$). This equivalent formulation holds for all dispositions of the requested eigenvalues $\lambda_1,\dots,\lambda_n$, except for finitely many.

In the second part, we use this characterization in terms of linear subspaces to show that the problem admits a solution when $m$, $p$ are specified as above ; the main technical tool is Borsuk's Antipodal Theorem.

The present problem does not, by any means, carry any difficulty which approaches, by far, that of the Invariant Subspace problem ; however it contains some technicalities which we will carefully handle. The main one is due to the fact that the eigenvalues to be assigned may be of two different kinds : either real, or complex non-real, pairwise conjugate.

Therefore, we advise the reader to go first through § 1, Part I, and § 1 and 2, Part II : they contain a detailed treatment of the simplest case : the assignment of a set of real numbers. All key ideas are already present there, and are much more transparent than in the general case. One also sees how the known case $m+p>n$ follows from our reduction, in a very simple manner.

We have not tried here to investigate the exceptional situations, except in Part I (where this is easy). How to recognize them, how to handle them, clearly deserves further work.

Many key topics will also deserve further study : how the assignment depends on the data (robustness), how behaves the operator norm of $K$, just to mention two of them.

We have not tried to find the best constants for small values of $n$. When $n$ is small, the computations made in Proposition 2.4 and Theorem 3.1 below can be made more accurate. As they are stated here, our results take full value only when $n$ is large : they say that the assignment is possible as soon as $\min(m,p) \geq \sqrt{8n}$, whereas the previously known result was $\min(m,p) > n/2$. Strictly speaking, this is an improvement only if $\sqrt{8n} < n/2$, that is $n > 32$, and some more work would be needed to improve our estimates for small values of $n$ also. But for large values of $n$, the improvement from $c \cdot n$ to $c \cdot \sqrt{n}$ is not only numerical ; it is also conceptual, because $\sqrt{n}$ is obviously best possible : one cannot go beyond.

Finally, we have not tried to build any numerical procedure : the second part of our proof, relying on Borsuk's theorem, is non-constructive. Borsuk's theorem is like the intermediate value theorem : it says that several functions, under certain conditions, have a common zero ; it does not say where it is. The same way, the intermediate value theorem will tell us that a polynomial with odd degree has a real zero, and then some numerical procedure has to be designed in order to find that zero. But trying to develop the numerical procedure *before* the problem has been analyzed at a theoretical level is just like moving a mass without inventing the wheel : slow, painful and costly. Clearly also, the wheel is a theoretical device, and further work is required before it can be used in practice. However, there will always be barbarians on horses claiming that the wheel is not necessary, and that they are happy with what they have.

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## Part I

## From Operator Theory to Intersection Properties of Linear Subspaces.

### 1. – A simple, generic, case.

We assume here that $m = p$ ; $F = \mathbb{R}^p$ is the subspace of $\mathbb{R}^n$ spanned by the first $p$ coordinates. $B$ : $\mathbb{R}^p \to \mathbb{R}^n$ is the canonical embedding (just denoted by J), and $C : \mathbb{R}^n \to \mathbb{R}^p$ is the orthogonal projection (denoted by P). The $\lambda_1, \ldots, \lambda_n$ to be assigned are real, pairwise distinct ($\lambda_i \neq \lambda_j$ if $i \neq j$) and none of them is in the spectrum of $A$, $\sigma(A)$.

Let $R(\lambda)$, $\lambda \notin \sigma(A)$, be the resolvent of $A$ : $R(\lambda) = (A - \lambda I)^{-1}$ (see B. Beauzamy [1] for basic facts about operator theory) ; let $F_1 = R(\lambda_1)F, \ldots, F_p = R(\lambda_p)F$ : these are $p$-dimensional subspaces of $\mathbb{R}^n$.

Let $N = Ker\ P$ (a $n-p$ dimensional subspace) and

$$N_{p+1} \;=\; (A-\lambda_{p+1}I)N, \ldots, N_n \;=\; (A-\lambda_n I)N,$$

these are $n-p$ dimensional subspaces.

For reasons which will be clear in the course of computations, we work with $A - JKP$ rather than with $A + JKP$.

**Theorem 1.1.** – *The assignment of $\sigma(A-JKP)$ to $\{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_1, \ldots, \lambda_n$ are real, pairwise distinct, and do not belong to $\sigma(A)$ is possible, by means of an isomorphism $K$, if and only if we can find a p-dimensional subspace $H$ of $\mathbb{R}^n$ which intersects (non-trivially) $F_1, \ldots, F_p$, at independent points $x_1, \ldots, x_p$, and also intersects $N_{p+1}, \ldots, N_n$.*

**Proof** : Let $\lambda_1, \ldots, \lambda_n$ be a $n$-tuple of distinct real numbers, none of them being in $\sigma(A)$. If $\sigma(A-JKP) = \{\lambda_1, \ldots, \lambda_n\}$, it means that we can find vectors $x_1, \ldots, x_n$ in $\mathbb{R}^n$, all not zero, such that :

$$(A-JKP)x_j \;=\; \lambda_j x_j \ , \qquad j = 1, \ldots, n, \tag{1.1}$$

or :

$$(A-\lambda_j I)x_j \;=\; JKPx_j \ , \qquad j = 1, \ldots, n. \tag{1.2}$$

Let $R(\lambda_j) = (A-\lambda_j I)^{-1}$, and set $y_j = (A-\lambda_j I)x_j$, or $x_j = R(\lambda_j)y_j$, $j = 1, \ldots, n$.

Equation (1.2) means that $y_j \in F$ $(j = 1, \ldots, n)$, and that in $F$ :

$$y_j \;=\; KPR(\lambda_j)y_j \ , \qquad j = 1, \ldots, n. \tag{1.3}$$

The key idea of the method is as follows : if we choose $p$ independent vectors $y_1, \ldots, y_p$ in $F$, the vectors $R(\lambda_j)y_j$ $(j = 1, \ldots, p)$ will also be independent in general and so will be the $PR(\lambda_j)y_j$, $j = 1, \ldots, p$. Then the $p$ equations (1.3) for $j = 1, \ldots, p$ determine $K$ completely, since $K$ is an operator from $F$ into $F$. The $n-p$ remaining equations (1.3) (for $j = p+1, \ldots, n$) will be treated as *compatibility conditions*, which we now write.

Assume $y_1, \ldots, y_p$ have been chosen, so as to be a basis of $F$. Then each of the vectors $y_{p+1}, \ldots, y_n$ admits a decomposition on this basis :

$$y_j \;=\; \alpha_1^{(j)} y_1 + \cdots + \alpha_p^{(j)} y_p \ , \qquad j = p+1, \ldots, n. \tag{1.4}$$

Then :

$$KPR(\lambda_j)y_j \;=\; \alpha_1^{(j)} KPR(\lambda_j)y_1 + \cdots + \alpha_p^{(j)} KPR(\lambda_j)y_p \ , \qquad j = p+1, \ldots, n. \tag{1.5}$$

So, the condition $y_j = KPR(\lambda_j)y_j$, $j = p+1, \ldots, n$ is equivalent, using (1.4), to :

$$\alpha_1^{(j)} y_1 + \cdots + \alpha_p^{(j)} y_p \;=\; \alpha_1^{(j)} KPR(\lambda_j)y_1 + \cdots + \alpha_p^{(j)} KPR(\lambda_j)y_p \ , \qquad j = p+1, \ldots, n. \tag{1.6}$$

Using the fact that $y_j = KPR(\lambda_j)y_j$ for $j = 1, \ldots, p$, we get, for $j = p+1, \ldots, n$,

$$\begin{aligned} &\alpha_1^{(j)} KPR(\lambda_1)y_1 + \cdots + \alpha_p^{(j)} KPR(\lambda_p)y_p \\ &\qquad\qquad = \; \alpha_1^{(j)} KPR(\lambda_j)y_1 + \cdots + \alpha_p^{(j)} KPR(\lambda_j)y_p \ . \end{aligned} \tag{1.7}$$

So our compatibility conditions are equivalent to the $n-p$ equations :

$$\alpha_1^{(j)} KP(R(\lambda_1)-R(\lambda_j))y_1+\cdots+\alpha_p^{(j)} KP(R(\lambda_p)-R(\lambda_j))y_p \;=\; 0, \tag{1.8}$$

for $j=p+1,\ldots,n$. Using the resolvent equation $R(\lambda_i)-R(\lambda_j)=(\lambda_i-\lambda_j)R(\lambda_i)R(\lambda_j)$, and replacing $(\lambda_i-\lambda_j)\alpha_i^{(j)}$ by $\alpha_i^{(j)}$ (which we may do, since we assumed $\lambda_i-\lambda_j\neq 0$), we get :

$$\alpha_1^{(j)} KPR(\lambda_j)R(\lambda_1)y_1+\cdots+\alpha_p^{(j)} KPR(\lambda_j)R(\lambda_p)y_p \;=\; 0, \qquad j=p+1,\ldots,n. \tag{1.9}$$

Since we will take for $K$ an isomorphism, equations (1.9) are equivalent to :

$$PR(\lambda_j)(\alpha_1^{(j)} R(\lambda_1)y_1+\cdots+\alpha_p^{(j)} R(\lambda_p)y_p) \;=\; 0, \qquad j=p+1,\ldots,n. \tag{1.10}$$

Set $H=\text{span}\{R(\lambda_1)y_1,\ldots,R(\lambda_p)y_p\}$, $N=Ker\ P$ (a $n-p$ dimensional subspace), and for $j=p+1,\ldots,n$ :

$$N_j \;=\; Ker\ PR(\lambda_j) \;=\; \{z,\ PR(\lambda_j)z=0\} \;=\; \{z,\ R(\lambda_j)z\in N\} \;=\; (A-\lambda_j I)N,$$

and $N_j$ is also a $n-p$ dimensional subspace, $j=p+1,\ldots,n$. Then, since we requested $R(\lambda_1)y_1,\ldots,R(\lambda_p)y_p$ to be independent, equations (1.10) just mean that $N_j$ intersects (non-trivially) $H$, for $j=p+1,\ldots,n$.

Let now $F_j=R(\lambda_j)F$, $j=1,\ldots,p$ : this is a $p$-dimensional subspace. If we find a $p$-dimensional subspace $H$ which intersects $F_j$, $j=1,\ldots,p$, it means that there are points $x_1,\ldots,x_p$ in $H$, and points $y_1,\ldots,y_p$ in $F$, such that $x_j=R(\lambda_j)y_j$, $j=1,\ldots,p$. If we request the $x_1,\ldots,x_p$ to be independent, $H$ is spanned by these points (since dim $H=p$), and the theorem is proved.

Of course, in general, when we find a $p$-dimensional subspace $H$ which intersects $F_1,\ldots,F_p$, the intersection points are independent and $K$ is automatically an isomorphism.

However, the assumption "independent" is necessary, as D. Grayson pointed out to us : if $n=4$ and $p=2$, it is always possible to find a 2-dimensional subspace of $\mathbb{R}^4$ which intersects four 2-dimensional subspaces. But if we take $A=0$, the general assignment is of course impossible : no matter what $K$ is, we will always have $\lambda_3=\lambda_4=0$. In the present case, the two subspaces $F_1$ and $F_2$ coincide, and $H$ must be equal to them.

We also see that the case $m+p>n$ (so $p>n/2$) has a simple solution. Take any vectors $x_1$ in $N_{p+1}$, $\ldots$, $x_{n-p}$ in $N_n$, and complete them in an arbitrary manner into $p$ independent vectors $x_1$, $\ldots$, $x_p$. Take $H=span\{x_1,\ldots,x_p\}$. Then $H$ intersects $N_{p+1}$, $\ldots$, $N_n$, and also intersects automatically $F_1$, $\ldots$, $F_p$ since their dimension is $p>n/2$.

**2. – The general case.**

We will first go through several simplifications, which we can make without loss of generality.

First, we may assume $p\le m$. Indeed, for any matrix $M$,

$$\sigma(M^*) \;=\; conj\{\sigma(M)\} \;=\; \{\overline{z},\ z\in\sigma(M)\}.$$

Since here all matrices have real entries,

$$(A+BKC)^* \;=\; {}^t(A+BKC) \;=\; {}^tA+\ {}^tC\ {}^tK\ {}^tB.$$

Since the set we want to assign is invariant under complex conjugation, we may replace $A+BKC$ by its transpose, that is we may assume $p\le m$, which we do in the sequel.

The second thing we observe is that $B$ and $C$ play very little role : only $Ker\ C$ and $Im\ B$ are important :

**Lemma 2.1.** – *The set of operators $BKC$, $K$ arbitrary $m \times p$ matrix, is the set of all operators from $\mathbb{R}^n / Ker\ C$ into $ImB$, with rank $p$.*

**Proof** of Lemma 2.1 – Let $\mathbb{R}^n = E = Ker\ C \oplus E_1$, where $\dim E_1 = p$, $\dim Ker\ C = n - p$. Then for any $K$, $BKC$ is an operator from $E_1$ into $Im\ B$.

Conversely, let $G$ be any $p$-dimensional subspace of $Im\ B$ (recall that dim $ImB = m \geq p$). Let $G_1 = B^{-1}(G)$, also $p$-dimensional.

Let $M$ be any operator from $E_1$ onto $G$ ; since $B$ is an isomorphism from $G_1$ onto $G$, and since $C$ is an isomorphism from $E_1$ onto $C(E_1)$, we can find $K$ such that $BKC = M$, and the lemma is proved.

We might therefore decide that $B$ is an injection and $C$ is a projection, but this would require a preliminary change of $K$. Since we want to stay with the original $K$, we will keep $B$ and $C$ as they are. We let (as in § 1) $N = Ker\ C$ (a $n - p$ dimensional space) and $F = Im\ B$ (a $m-$dimensional subspace).

So far, we have not met anything significantly different from the simplified case.

We now turn to the study of non-real eigenvalues.

Our self-conjugate set $S$ of $n$ complex numbers, to be assigned, will be numbered as follows (where $0 \leq k \leq n/2$)

– first $n - 2k$ real numbers, numbered $\lambda_1, \ldots, \lambda_{n-2k}$,
– last, $2k$ non real numbers. We let $\lambda_{n-2k+1}, \ldots, \lambda_{n-k}$ be the ones with $Im\lambda > 0$, and we do not number their conjugates.

This indexation may seem strange, since $\lambda_{n-k+1}, \ldots, \lambda_n$ do not exist explicitly ; it will however prove efficient in the sequel. Introducing the $\overline{\lambda}_i' s$ in the indexation leads instead to redundant equations and must be avoided.

If $k = 0$, all eigenvalues are real (this is the simplified case seen above) ; if $n$ is even and $k = n/2$, none is real.

We will say that the set $S$ to be assigned is *essentially real* if $k \leq p/2$ (thus $n - 2k \geq n - p$). We will say that it is *essentially complex* (one should of course say "essentially non-real") if $k > p/2$.

This distinction will be used for clarity. We pick up $p$ "principal" equations in the essentially real case, $p/2$ in the essentially complex one, and write the others as compatibility equations. Distinguishing between both cases will lead to a clearer presentation of the statements and of the proofs.

In the sequel, we restrict ourselves to the case $p \leq n/2$ : on one hand, it is always possible to diminish $p$ by putting some zeros in $K$ (and the smaller $p$ is, the harder the assignment will be) ; on the other hand, the case $p > n/2$ is already known.

If $H$ is a real subspace of $\mathbb{R}^n$, say $H = span\{z_1, \ldots, z_p\}$ for some basis $\{z_1, \ldots, z_p\}$, the associated (or underlying) complex space, denoted by $H_{\mathbb{C}}$, is simply

$$H_{\mathbb{C}} = \left\{ \sum_1^n \alpha_j z_j \ ; \ \alpha_1, \ldots, \alpha_n \in \mathbb{C} \right\}.$$

This is independent of the basis, and $\overline{H_{\mathbb{C}}} = H_{\mathbb{C}}$.

We may now state the general intersection theorem.

**Theorem 2.2.** – *Let $n$, $p$, $m$, with $p \leq m$ and $p \leq n/2$, and let $S$ be a self-conjugate set of $n$ complex numbers, pairwise distinct, none of them being in $\sigma(A)$.*

*1) If the set $S$ is essentially real, the assignment problem is equivalent to the existence of a subspace $H$ of $\mathbb{R}^n$, satisfying simultaneously :*

*a) $H$ intersects the $p$ real subspaces $R(\lambda_1)F, \ldots, R(\lambda_p)F$ (all of dimension $m$), and is spanned by these intersections,*

*b) $H$ intersects the $n-2k-p$ real subspaces $(A-\lambda_{p+1}I)N, \ldots, (A-\lambda_{n-2k}I)N$ (all of dimension $n-p$),*

*c) the associated complex space $H_{\mathbb{C}}$ intersects $(A-\lambda_{n-2k+1}I)N, \ldots, (A-\lambda_{n-k}I)N$ (all of complex dimension $n-p$).*

*2) If the set $S$ is essentially complex, let $\mu = p/2$ if $p$ is even, $(p+1)/2$ if $p$ is odd. The assignment problem is equivalent to the existence of a $\mu$-dimensional space $H$ of $\mathbb{C}^n$, satisfying simultaneously :*

*a) $H$ intersects the $\mu$ complex subspaces $R(\lambda_{n-k-\mu+1})F_{\mathbb{C}}, \ldots, R(\lambda_{n-k})F_{\mathbb{C}}$ (all of dimension $m$), and is spanned by these intersections,*

*b) $Re(H)$ (the real parts of vectors in $H$) intersects the real subspaces $(A-\lambda_1 I)N, \ldots, (A-\lambda_{n-2k}I)N$ (all of dimension $n-p$),*

*c) $H \oplus \overline{H}$ intersects the complex subspaces $(A-\lambda_{n-2k+1}I)N, \ldots, (A-\lambda_{n-k-\mu}I)N$*

(all of dimension $n-p$).

We now turn to the proof of the theorem. As before, we prefer to work with $A-BKC$. To say that

$$\sigma(A-BKC) = \{\lambda_1, \ldots, \lambda_{n-2k}, \lambda_{n-2k+1}, \ldots, \lambda_{n-k}, \overline{\lambda}_{n-2k+1}, \ldots, \overline{\lambda}_{n-k}\}$$

means that we can find vectors $z_1, \ldots, z_{n-2k} \in \mathbb{R}^n$, $z_{n-2k+1}, \ldots, z_{n-k} \in \mathbb{C}^n$, all non-zero, such that

$$(A-BKC)z_j = \lambda_j z_j , \qquad j = 1, \ldots, n-k. \tag{2.1}$$

Of course, for $\overline{\lambda}_j$ $(j = n-2k+1, \ldots, n-k)$, the eigenvector is automatically $\overline{z}_j$, and these equations may be omitted.

Equations (2.1) can be written :

$$(A-\lambda_j I)z_j = BKCz_j , \qquad j = 1, \ldots, n-k. \tag{2.2}$$

Set $y_j = (A-\lambda_j I)z_j$, or $z_j = R(\lambda_j)y_j$. Then $y_j \in \mathbb{R}^n$ for $j \leq n-2k$, $y_j \in \mathbb{C}^n$, $j \geq n-2k+1$, and (2.2) becomes :

$$y_j = BKC\, R(\lambda_j)y_j , \qquad j = 1, \ldots, n, \tag{2.3}$$

with $\lambda_{n-k+1} = \overline{\lambda}_{n-2k+1}, \ldots, \lambda_n = \overline{\lambda}_{n-k}$, $y_{n-k+1} = \overline{y}_{n-2k+1}, \ldots, y_n = \overline{y}_{n-k}$.

The next lemma says there is no need to worry about the matrix $K$ being real : this will be automatic.

**Lemma 2.3.** – *If the matrix $K$ satisfies the $n$ equations (2.3), and if the rank of the vectors $y_j$ is at least $p$, then $K$ is real.*

**Proof** of lemma 2.3 – The matrix $K : \mathbb{C}^p \to \mathbb{C}^m$ is totally determined if we know the images of $p$ independent vectors, that is $K$ is uniquely determined by (2.3) if we assume the rank of the $y_j$'s to be at least $p$. But $\overline{K}$ satisfies the same equations, and therefore $K = \overline{K}$.

We now distinguish between the essentially real and the essentially complex cases.

**A. – The essentially real case**.

Since $n-2k \geq n-p \geq p$, at least $p$ of the equations (2.3) are real ones. We take the first $p$ equations (2.3) as "principal" ones, and write the others as compatibility conditions.

We take $y_j$ , $j>p$, under the form :

$$y_j \;=\; \alpha_1^{(j)} + \cdots + \alpha_p^{(j)} y_p \;, \tag{2.4}$$

with $\alpha_1^{(j)}, \ldots, \alpha_p^{(j)}$ real if $j = p+1, \ldots, n-2k$, and $\alpha_1^{(j)}, \ldots, \alpha_p^{(j)}$ complex if $j > n-2k$. Of course, for each $j$, at least one of the $\alpha_i^{(j)}$ $(i = 1, \ldots, p)$ must be non-zero.

Exactly the same computations as in § 1 lead to :

$$BKC\; R(\lambda_j) \left(\alpha_1^{(j)} R(\lambda_1) y_1 + \cdots + \alpha_p^{(j)} R(\lambda_p) y_p\right) \;=\; 0 \;, \qquad j > p. \tag{2.5}$$

Since $B$ is an isomorphism into its image, and since we will take for $K$ an isomophism, equations (2.5) become :

$$C\; R(\lambda_j) \left(\alpha_1^{(j)} R(\lambda_1) y_1 + \cdots + \alpha_p^{(j)} R(\lambda_p) y_p\right) \;=\; 0 \;, \qquad j > p. \tag{2.6}$$

We observe that (2.6) is stronger than (2.5), even if $K$ is not an isomorphism.

Let's now prove Theorem 2.1, 1).

Assume we have found a $p$-dimensional subspace $H$ of $\mathbb{R}^n$, satisfiying a), b) and c). Since $H$ intersects $R(\lambda_1)F, \ldots, R(\lambda_p)F$, there are points $z_j$ in $H$, points $y_j$ in $F$ $(j = 1, \ldots, p)$, such that

$$z_j \;=\; R(\lambda_j) y_j \;, \qquad j = 1, \ldots, p, \tag{2.7}$$

and $H = span\{z_1, \ldots, z_p\}$.

Since $H$ intersects the $n-2k-p$ real subspaces $(A-\lambda_{p+1}I)N, \ldots, (A-\lambda_{n-2k}I)N$, we can find real linear combinations $\alpha_1^{(j)} z_1 + \cdots + \alpha_p^{(j)} z_p$ ($\alpha_1^{(j)}, \ldots,\ \alpha_p^{(j)}$ real) with

$$\alpha_1^{(j)} z_1 + \cdots + \alpha_p^{(j)} z_p \;\in\; (A-\lambda_j I)N, \qquad j = p+1, \ldots, n-2k,$$

which means (2.6) for $j = p+1, \ldots, n-2k$.

Since $H_{\mathbb{C}}$ intersects the complex subspaces $(A-\lambda_{n-2k+1}I)N, \ldots, (A-\lambda_{n-k}I)N$, we can find complex linear combinations $\alpha_1^{(j)} z_1 + \cdots + \alpha_p^{(j)} z_p$ ($\alpha_1^{(j)}, \ldots, \alpha_p^{(j)}$ complex) with

$$\alpha_1^{(j)} z_1 + \cdots + \alpha_p^{(j)} z_p \;\in\; (A-\lambda_j I)N, \qquad j = n-2k+1, \ldots, n-k,$$

and this means (2.6) for $j = n-2k+1, \ldots, n-k$.

Conversely, assume that the assignment can be realized. It means that $y_1, \ldots, y_p$ can be found, and, for each $j > p$, a collection $(\alpha_1^{(j)}, \ldots, \alpha_p^{(j)})$ such that (2.6) holds. Let $H = span\{R(\lambda_1)y_1, \ldots, R(\lambda_1)y_p\}$. Obviously, a) is satisfied, and (2.6) implies b) and c).

So Theorem 2.2 is proved in this case.

**B. – The essentially complex case.**

We now have a large supply of complex equations (2.3) : the number of such equations, with $Im\lambda > 0$, is $k > p/2$.

If $\mu = p/2$ ($p$ even) or $(p+1)/2$ ($p$ odd), at least $\mu$ such equations can be found. They are the last $\mu$ equations in the list, and correspond to the labels $\lambda_{n-k-\mu+1}, \ldots, \lambda_{n-k}$, with corresponding $y_j$'s.

In order to simplify the notation, we let

$$\lambda'_1 \;=\; \lambda_{n-k-\mu+1}, \ldots, \lambda'_\mu \;=\; \lambda_{n-k} \;,$$

$$y'_1 \;=\; y_{n-k-\mu+1}, \ldots, y'_\mu \;=\; y_{n-k}.$$

The other $y_j$'s ($j \leq n-k-\mu$) will be taken under the form :

$$y_j \;=\; \alpha_1^{(j)} y'_1 + \beta_1^{(j)} \overline{y}'_1 + \cdots + \alpha_\mu^{(j)} y'_\mu + \beta_\mu^{(j)} \overline{y}'_\mu \;. \tag{2.8}$$

If $j \leq n-2k$, $y_j$ is real, and we take moreover : $\beta_1^{(j)} = \overline{\alpha}_1^{(j)}, \ldots, \beta_\mu^{(j)} = \overline{\alpha}_\mu^{(j)}$.

Exactly the same computations as in § 1 now give, for $j \leq n-k-\mu$ :

$$CR(\lambda_j)\left(\alpha_1^{(j)} R(\lambda'_1)y'_1 + \beta_1^{(j)} R(\overline{\lambda}'_1)\overline{y}'_1 + \cdots + \alpha_\mu^{(j)} R(\lambda'_\mu)y'_\mu + \beta_\mu^{(j)} R(\overline{\lambda}'_\mu)\overline{y}'_\mu\right) \;=\; 0. \tag{2.9}$$

If $j \leq n-2k$, this becomes simply :

$$CR(\lambda_j)\left(Re\sum_{i=1}^{n} \alpha_i^{(j)} R(\lambda'_i)y'_i\right) \;=\; 0. \tag{2.10}$$

Assume now we have found a subspace $H$ of $\mathbb{C}^n$ satisfying a), b), c). Since $H$ intersects the subspaces $R(\lambda'_1)F_{\mathbb{C}}, \ldots, R(\lambda'_\mu)F_{\mathbb{C}}$, there exist points $z'_1, \ldots, z'_\mu$ in $H$, $y'_1, \ldots, y'_\mu$ in $F_{\mathbb{C}}$ (all non-zero) such that

$$z'_j \;=\; R(\lambda'_j)y'_j \;, \qquad j = 1, \ldots, \mu, \tag{2.11}$$

and $H = span_{\mathbb{C}}\{z'_1, \ldots, z'_\mu\}$.

Equations (2.10) mean that $Re(H)$ intersects $(A-\lambda_j I)N$, $j = 1, \ldots, n-2k$, and equations (2.9) mean that $H \oplus \overline{H}$ intersects $(A-\lambda_j I)N$, $j = n-2k+1, \ldots, n-k-\mu$. So the assignment can be realized. The converse is obvious, and the theorem is proved.

**Part II.– Intersecting linear subspaces.**

We have seen in the first part (Theorem 2.2) that the assignment of $\{\lambda_1, \ldots, \lambda_n\}$ (either real or pair-wise conjugate), not belonging to $\sigma(A)$, was equivalent to an intersection problem for linear subspaces. We now describe the required intersection properties by means of linear equations.

**1. Representing linear subspaces by means of linear operators.**

Let $F$ be a $m$-dimensional subspace of $\mathbb{R}^n$. If we write any $z \in \mathbb{R}^n$ as $z = (z', z'')$, where $z' \in \mathbb{R}^m$, $z'' \in \mathbb{R}^{n-m}$, we may describe $F$ as :

$$F \;=\; \{z \in \mathbb{R}^n \;;\; z'' \;=\; Tz'\}, \tag{1.1}$$

for some operator $T : \mathbb{R}^m \to \mathbb{R}^{n-m}$.

The representation is not true for <u>all</u> subspaces of $\mathbb{R}^n$ : only for those which have no linear dependence among the first $m$ coordinates ; such subspaces will be called "in general position". But this is not a restriction, as we will see later.

Similarly, if $N$ is a $n-p$ dimensional subspace of $\mathbb{R}^n$, there is an operator $V : \mathbb{R}^{n-p} \to \mathbb{R}^p$ such that

$$N = \{z \in \mathbb{R}^n \ ; \ z' = Vz''\}, \tag{1.2}$$

when $N$ is in general position (that is, this time, when there is no linear dependence between the last $n-p$ coordinates).

Let's first observe that to be in general position is not a restriction : one should simply choose correctly the basis of $\mathbb{R}^n$.

**Lemma 1.1.** – *Given $p$ subspaces $F_1, \ldots, F_p$ of the space $\mathbb{R}^n$ with dimension $m < n$ and $n-p$ subspaces $N_{p+1}, \ldots, N_n$ of $\mathbb{R}^n$, with dimension $n-p$, one can always find a basis of $\mathbb{R}^n$ such that all these subspaces are in general position.*

**Proof** of Lemma 1.1. – First, take any non-zero vectors $f_1^{(1)}, \ldots, f_p^{(1)}$ in $F_1, \ldots, F_p$ respectively, and any non-zero vectors $g_{p+1}^{(1)}, \ldots, g_n^{(1)}$ in $N_{p+1}, \ldots, N_n$ respectively. Let $H_1$ be a hyperplane which contains none of these vectors, and let $e_1 \perp H_1$. The vector $e_1$ will be the first vector of our basis, and all further vectors will be chosen in $H_1$. It is clear that the $f_j^{(1)}$'s and the $g_j^{(1)}$'s have a non-zero composent on $e_1$. Now each $F_j \cap H_1$ is exactly $m-1$ dimensional, and each $N_j \cap H_1$ is exactly $n-p-1$ dimensional.

Take now any non-zero vectors $f_1^{(2)}, \ldots, f_p^{(2)}$ in $F_1 \cap H_1, \ \ldots, \ F_p \cap H_1$ respectively and any non-zero vectors $g_{p+1}^{(2)}, \ldots, g_n^{(2)}$ in $N_{p+1} \cap H_1, \ldots, \ N_n \cap H_1$ ; let $H_2$ be a hyperplane of $H_1$, not containing any of the $f_j^{(2)}$'s or any of the $g_j^{(2)}$'s, and let $e_2 \in H_1$, $e_2 \perp H_2$, and so on. The constructed basis has the required property.

The above construction of the basis $(e_j)_{j=1,\ldots,n}$ is not critical and, in fact, if one chooses at random vectors $e_1, \ldots, e_n$ in the unit ball of $\mathbb{R}^n$, they will (with probability 1) form a basis with respect to which the fixed subspaces $F_1, \ldots, F_p$, $N_{p+1}, \ldots, N_n$ will be in general position.

What we said for subspaces of $\mathbb{R}^n$ applies also, obviously, to subspaces of $\mathbb{C}^n$, without any modification.

In the sequel, we assume that the basis of $\mathbb{R}^n$ or $\mathbb{C}^n$ has been properly chosen (with respect to the required assignment) and that the subspaces we meet are in general position.

As we already did in the First Part, we study in detail the case of the assignment of $n$ real numbers. This is of course a special case, but the techniques will be more transparent. We will then turn to the general cases : the essentially real case and the essentially complex one.

## 2. A special case : the assignment of $n$ real numbers.

We assume as before $p \le m \le n$, $p \le n/2$.

**Theorem 2.1.** – *The assignment of $\lambda_1, \ldots, \lambda_n$, all real, distinct and not in $\sigma(A)$ is possible in general if*

$$\begin{aligned} p &\ge \sqrt{8n+33} - 5, && \text{when } n-p \text{ is odd,} \\ p &\ge \sqrt{8n+33} - 4, && \text{when } n-p \text{ is even.} \end{aligned}$$

**Proof** of Theorem 2.1. – Since our estimates use only $\min(p,m)$, we may assume $m = p$, which we do for simplicity. We now divide the proof into sections, which will be used later.

**1.a. – Notations**.

Let now $F_j = R(\lambda_j)F$ $(j = 1, \ldots, p)$ be the $p$ subspaces of dimension $p$, and $N_j = (A - \lambda_j I)N$ $(j = p+1, \ldots, n)$ be the $n-p$ subspaces of dimension $n-p$ defined in Part I, Theorem 1.1, or Theorem 2.2 (in this case, $k = 0$ : all real).

According to § 1, we write

$$F_j \;=\; \{z \in \mathbb{R}^n \;;\; z'' \;=\; T_j z'\}, \tag{2.a.1}$$

where each $T_j$ is a linear operator $\mathbb{R}^p \;\to \mathbb{R}^{n-p}$, $j = 1, \ldots, p$.

Similarly, we write

$$N_j \;=\; \{z \in \mathbb{R}^n \;;\; z' \;=\; V_j z''\}, \tag{2.a.2}$$

where each $V_j$ is a linear operator $\mathbb{R}^{n-p} \;\to \mathbb{R}^p$, $j = p+1, \ldots, n$.

**2.b. – From linear operators to linear equations**.

We want to find a subspace $H$, intersecting $F_1, \ldots, F_p$, spanned by these intersections (thus $H$ will be $p$-dimensional) and also intersecting $N_{p+1}, \ldots, N_n$.

The subspace $H$ will be determined by $p$ independent vectors $u_1, \ldots, u_p$ in $\mathbb{R}^n$. The fact that $H$ intersects $F_1$ non-trivially means that there is a non-zero linear combination $a_{1,1}u_1 + \cdots + a_{1,p}u_p$, belonging to $F_1$, that is, by (2.a.1), satisfying

$$T_1(a_{1,1}u'_1 + \cdots + a_{1,p}u'_p) \;=\; a_{1,1}u''_1 + \cdots + a_{1,p}u''_p,$$

and since the $u_j$'s are independent, in order to ensure that this combination is non-zero, all we need is that one of the $a_{1,j}$ $(j = 1, \ldots, p)$ should be non-zero.

We argue the same way with $F_2, \ldots, F_p$, $N_{p+1}, \ldots, N_n$ and we obtain :

**Proposition 2.2.** – *The assignment of $\lambda_1, \ldots, \lambda_n$, real, distinct and not in $\sigma(A)$ can be realized if and only if there are independent vectors $u_1, \ldots, u_p$ in $\mathbb{R}^n$, such that, if $u = (u', u'')$, $u' \in \mathbb{R}^p$, $u'' \in \mathbb{R}^{n-p}$, the following two sets of equations are simultaneously satisfied :*

$$\begin{cases} T_1(a_{1,1}u'_1 + \cdots + a_{1,p}u'_p) &= a_{1,1}u''_1 + \cdots + a_{1,p}u''_p \\ & \vdots \\ T_p(a_{p,1}u'_1 + \cdots + a_{p,p}u'_p) &= a_{p,1}u''_1 + \cdots + a_{p,p}u''_p \end{cases} \tag{2.b.1}$$

$$\begin{cases} a_{p+1,1}u'_1 + \cdots + a_{p+1,p}u'_p &= V_{p+1}(a_{p+1,1}u''_1 + \cdots + a_{p+1,p}u''_p) \\ & \vdots \\ a_{n,1}u'_1 + \cdots + a_{n,p}u'_p &= V_n(a_{n,1}u''_1 + \cdots + a_{n,p}u''_p) \end{cases} \tag{2.b.2}$$

where in each of these $n$ equations at least one of the $a_{i,j}$'s must be non-zero.

**2.c. – From equations in $\mathbb{R}^n$ to equations in $\mathbb{R}^{n-p}$.**

In all the above equations, appear both $u'_j$ and $u''_j$. We use the last $p$ equations of (2.b.2) in order to compute (in a trivial manner) each $u'_j$ from the corresponding $u''_j$. In the last equation (2.b.2), we set $a_{n,p} = 1$, all other $a_{n,i} = 0$, and get

$$u'_p \;=\; V_n\, u''_p. \tag{2.c.1}$$

Similarly, in the previous one, we set $a_{n-1,p-1} = 1$, all other $0$, and get

$$u'_{p-1} \;=\; V_{n-1}\; u''_{p-1}\;,$$

and proceed the same way until

$$u'_1 \;=\; V_{n-p+1}\; u''_1.$$

Substituting the $u'_j$ into the $p$ equations (2.b.1) and the remaining $n-2p$ equations (2.b.2) gives :

$$\begin{cases} a_{1,1}(T_1V_{n-p+1}-I)u''_1+\cdots+a_{1,p}(T_1V_n-I)u''_p &= 0\\ &\vdots\\ a_{p,1}(T_pV_{n-p+1}-I)u''_1+\cdots+a_{p,p}(T_pV_n-I)u''_p &= 0\end{cases} \tag{2.c.2}$$

and

$$\begin{cases} a_{p+1,1}(V_{n-p+1}-V_{p+1})u''_1+\cdots+a_{p+1,p}(V_n-V_{p+1})u''_p &= 0\\ &\vdots\\ a_{n-p,1}(V_{n-p+1}-V_{n-p})u''_1+\cdots+a_{n-p,p}(V_n-V_{n-p})u''_p &= 0\end{cases} \tag{2.c.3}$$

Now, in order to simplify our notation, we set $x_j = u''_j \in \mathbb{R}^p$, $j = 1,\ldots,p$, and $W_{i,j} = V_{n-p+j} - V_i$, $i = p+1,\ldots,n-p$, $j = 1,\ldots,p$.

So $W_{i,j} : \mathbb{R}^{n-p} \to \mathbb{R}^p$ for each $i,j$. Equations (2.c.3) become :

$$\begin{cases} a_{p+1,1}W_{p+1,1}x_1+\cdots+a_{p+1,p}W_{p+1,p}x_p &= 0\\ &\vdots\\ a_{n-p,1}W_{n-p,1}x_1+\cdots+a_{n-p,p}W_{n-p,p}x_p &= 0\end{cases} \tag{2.c.4}$$

We now turn to the study of equations (2.c.4). Just one $x_j$ will be enough to take care of them.

**2.d. – Killing many animals with a single bullet**.

**Proposition 2.3.** – *For every choice of* $x_1,\ldots,x_{p-1} \neq 0$, *there exists a non-zero* $x_p$ *and a choice of* $a_{i,j}$ $(i = p+1,\ldots,n-p,\; j = 1,\ldots,p)$, *with a least one* $a_{i,j} \neq 0$ *for each* $i$, *such that all equations (2.c.4) are simultaneously satisfied.*

**Proof** of Proposition 2.3. – Fix $x_1,\ldots,x_{p-1} \neq 0$. Consider the application $f$, from the unit ball of $\mathbb{R}^{n-p}$ into $\mathbb{R}^{n-2p}$ :

$$\begin{aligned} f(x) \;=\; \big(&det(W_{p+1,1}x_1,\ldots,W_{p+1,p-1}x_{p-1},W_{p+1,p}x),\\ &\vdots\\ &det(W_{n-p,1}x_1,\ldots,W_{n-p,p-1}x_{p-1},W_{n-p,p}x)\big).\end{aligned}$$

It satisfies $f(-x) = -f(x)$ and so by Borsuk's antipodal theorem, there is a $x$, $\|x\| = 1$, such that $f(0) = 0$.

This implies that all determinants are zero, and proves the proposition.

**Remark**. – There is a more direct way to handle the equations (2.b.2), which does not require the substitutions (2.c.1) and leaves room for more freedom in the choice of the $u'_j$. Indeed, instead of $f(x)$ defined above, consider :

$$g(u) \;=\; \Big(det(u'_1 - V_{p+1}u''_1, \ldots, u'_{p-1} - V_{p+1}u''_{p-1}, u' - V_{p+1}u''),$$
$$\vdots$$
$$det(u'_1 - V_n u''_1, \ldots, u'_{p-1} - V_n u''_{p-1}, u' - V_n u'')\Big),$$

where $u \in \mathbb{R}^n$, $\|u\| = 1$. This is an odd function of $u$, with values in $\mathbb{R}^{n-p}$, and Borsuk applies.

This method leads to the same estimates, but does not require any link between $u'_j$ and $u''_j$, $j = 1, \ldots, p-1$. So we may decide to fix arbitrary operators $S_1, \ldots, S_{p-1} : \mathbb{R}^{n-p} \to \mathbb{R}^p$, and set $u'_j = S_j u''_j$, $j = 1, \ldots, p-1$. This method will be used in § 3 and 4 below, and may also prove useful for an investigation of the exceptional cases.

We now turn to equations (2.c.2), which are much harder to solve, because they are in $\mathbb{R}^{n-p}$ and not in $\mathbb{R}^p$.

**2.e. – Solving equations in $\mathbb{R}^{n-p}$.**

In equations (2.c.2), we take $a_{1,p} = \cdots = a_{p,p} = 0$, so $x_p$ will not appear in them. We also introduce the notation

$$U_{i,j} \;=\; T_i V_{n-p+j} - I, \qquad i = 1, \ldots, p-1,\; j = 1, \ldots, p.$$

So equations (2.c.2) become :

$$\begin{cases} a_{1,1}U_{1,1}x_1 + \cdots + a_{1,p-1}U_{1,p-1}x_{p-1} \;= 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\vdots \\ a_{p,1}U_{p,1}x_1 + \cdots + a_{p,p-1}U_{p,p-1}x_{p-1} \;= 0, \end{cases} \tag{2.e.1}$$

where $U_{i,j} : \mathbb{R}^{n-p} \to \mathbb{R}^{n-p}$, and $x_1, \ldots, x_{p-1} \in \mathbb{R}^{n-p}$.

The idea of the solution is as follows : from the last equations (2.e.1) one computes the last variables $x_j$ in terms of the first ones. Then, one substitutes into the first equations, in such a way that one gets equations with exactly $n-p$ terms, in which the variables $x_j$ will be repeated but the $a_{i,j}$ are independent from one equation to the next. One then applies Borsuk's theorem to solve these "long" equations ; the remaining "short" ones are solved in a trivial manner. Unfortunately, since Borsuk's theorem requires the function to be odd, we will need to distinguish between the case $n-p$ odd and the case $n-p$ even, with minor differences.

**Proposition 2.4.** – *If $p \ge \sqrt{8n+33} - 5$ (case $n-p$ odd) or $p \ge \sqrt{8n+33} - 4$ (case $n-p$ even), the equations (2.e.1) admit a solution in general : there exist independent $x_i$'s and $(a_{i,j})$ (one at least non-zero for each $i$) satisfying (2.e.1).*

**Proof** of Proposition 2.4. – Let $l$, $h$ be two integers, such that $l.h < p$, to be specified later. Let $q = lh$.

Consider the last $q$ equations (2.e.1). From the last equation, compute $x_{p-1}$ in terms of $x_1, \ldots, x_{p-q-1}$ in a trivial manner :

$$U_{p,p-1}x_{p-1} \;=\; -(a_{p,1}U_{p,1}x_1 + \cdots + a_{p,p-q-1}U_{p,p-q-1}x_{p-q-1}). \tag{2.e.2}$$

We claim that $U_{p,p-1}$ is invertible in general. Indeed, $U_{p,p-1} = T_p V_{n-1} - I$. If some $x \ne 0$, $x \in \mathbb{R}^{n-p}$, satisfied $T_p V_{n-1} x = x$, then set $u'' = x$, $u' = V_{n-1}x \in \mathbb{R}^p$. Then $T_p u' = u''$ and $V_{n-1}u'' = u'$, which means that the vector $u = (u', u'') \in \mathbb{R}^n$, $u \ne 0$, would be both in $F_p$ and in $N_{n-1}$ : these two vector spaces would already intersect.

Similarly, we compute $x_{p-2}$ :

$$U_{p-1,p-2}x_{p-2} \;=\; -(a_{p-1,1}U_{p-1,1}x_1 + \cdots + a_{p-1,p-q-1}U_{p-1,p-q-1}x_{p-q-1}),$$

and so on, until

$$U_{p-q,p-q-1}x_{p-q} \;=\; -(a_{p-q,1}U_{p-q,1}x_1 + \cdots + a_{p-q,p-q-1}U_{p-1,p-q-1}x_{p-q-1}).$$

We observe that all these equations carry independent $a_{i,j}$'s. Let's now arrange the last $q$ variables $x_{p-q},\dots,x_{p-1}$ into $l$ blocks of length $h$ :

$$L_1 \;=\; (x_{p-q},\dots,x_{p-q-h+1}),\dots,L_l \;=\; (x_{p-h},\dots,x_{p-1}). \tag{2.e.3}$$

Let also $L_0$ be the block of the first variables :

$$L_0 \;=\; (x_1,\dots,x_{p-q-1}).$$

Now, the substitution process goes as follows :

– in the first equation (2.e.1), do the following
  * do not touch variables in $L_0$,
  * kill all variables except in $L_0$ and $L_l$ (that is, take the corresponding $a_{i,j}=0$)
  * substitute all variables in $L_l$, using (2.e.2).

Each substitution replaces one variable by $p-q-1$ variables, so the new equation has $(h+1)(p-q-1)$ variables $x_j$. These variables are not distinct : each of them is repeated $h+1$ times, but all the $a$'s are independent, and none of them will appear in any latter equation.

– in the second equation (2.e.1), do the following :
  * do not touch variables in $L_0$,
  * kill all variables except in $L_0$ and $L_{l-1}$,
  * substitute all variables in $L_{l-1}$, using (2.e.2).

– repeat this process $l$ times.

This way, we get $l$ "long" equations, involving $(h+1)(p-q-1)$ terms with independent $a$'s.

After we have used $q$ last equations to compute the last $x_j$'s from the first, and performed substitutions in the $l$ first equations, the number of "middle" equations left is of course $p-q-l$.

We want to keep $x_1$ for further use, in order to solve the long equations. The middle ones will be solved in a trivial manner, just reducing to two variables. The first one becomes :

$$a_{l+1,1}\;U_{l+1,1}\;x_1 + a_{l+1,2}\;U_{l+1,2}\;x_2 \;=\; 0,$$

and it can be solved in terms of $x_2$ : if $U_{l+1,2}$ is invertible, take $a_{l+1,1}=a_{l+1,2}=1$, if not, take $x_2$ in its kernel and $a_{l+1,1}=0$, $a_{l+1,2}=1$.

This trivial solving process for the middle equations requires one variable (after $x_1$) for each, that is, we need $p-q-l$ variables after $x_1$. Since the number of variables at our disposal is $p-q-2$ (after $x_1$), we find the condition

$$p-q-2 \;\geq\; p-q-l,$$

or

$$l \;\geq\; 2, \tag{\textit{cond} 1}$$

in order to solve the middle equations. The solution of these middle equations is in general expressed in linear terms :

$$\begin{cases} x_2 &= A_2 x_1 \\ x_3 &= A_3 x_1 \\ & \vdots \\ x_{p-q-1} &= A_{p-q-1} x_1 \end{cases} \tag{2.e.4}$$

We substitute all these variables in the $l$ first long equations, and, if any free variable remains, we take it also equal to $x_1$.

Now, we require that the length of the long equations is at least $n-p$, that is :

$$(h+1)(p-q-1) \ \geq \ n-p. \tag{cond 2}$$

If this length exceeds $n-p$, we kill the corresponding terms, by taking all the corresponding $a$'s to be zero. This way, we obtain a set of equations (with $x = x_1$)

$$\begin{cases} t_{1,1}A_{1,1}x + \cdots + t_{1,n-p}A_{1,n-p}x &= 0 \\ & \vdots \\ t_{l,1}A_{l,1}x + \cdots + t_{l,n-p}A_{l,n-p}x &= 0, \end{cases} \tag{2.e.5}$$

where the $A_{i,j}$ are given operators $\mathbb{R}^{n-p} \to \mathbb{R}^{n-p}$, $x \in \mathbb{R}^{n-p}$ is at our disposal, and so are the real scalars $t_{i,j}$. At least one of each $t_{i,j}$ should be non-zero in each line.

We consider the application $\varphi$, from the unit ball of $\mathbb{R}^{n-p}$ into $\mathbb{R}^l$ :

$$\varphi(x) \ = \ (det(A_{1,1}x, \ldots, A_{1,n-p}x), \ldots, det(A_{l,1}x, \ldots, A_{l,n-p}x)) \, .$$

When $n-p$ is odd, this is an odd function, and so by Borsuk's theorem there is an $x$, $\|x\| = 1$, such that $\varphi(x) = 0$. This solves all equations (2.e.5), provided conditions (cond 1) and (cond 2) are satisfied, in the case $n-p$ odd.

If $n-p$ is even, then we just drop one unit for $p$ from the original problem : we fix arbitrarily the last row and the last column in $K$, that is we consider it as a $(p-1) \times (p-1)$ matrix.

One can also decide, in order to handle this case, to keep one free variable, say $x_1$ : $x_{p-q}, \ldots, x_{p-1}$ are expressed in terms of $x_2, \ldots, x_{p-q-1}$ only, and $x_1$ is kept to apply Borsuk. This leads to weaker estimates.

We still have to study conditions (cond 1) and (cond 2). But before we do this, let's understand the structure of the proof.

The points $x_2, \ldots, x_{p-q-1}$ are determined from $x_1$ by (2.e.4). The variable $x = x_1$ itself is solution of (several) equations of the type :

$$a_1 U_1 x + a_2 U_2 A_2 x + \cdots + a_{p-q-1} U_{p-q-1} A_{p-q-1} x + a_{p-q} U_{p-q} x + a_{p-q+1} U_{p-q+1} A_2 x + \cdots \ = \ 0$$

(the $U$'s are different, the $A$'s repeat themselves).

Since the $U$'s are arbitrary operators, the points $x_1$, $A_2 x_1$,..., $A_{p-q-1} x_1$ will be linearly independent in general.

Now $x_{p-q},\ldots,\ x_{p-1}$ are determined from $x_1,\ldots,x_{p-q-1}$ by other independent operators, so the points $x_1,\ \ldots,\ x_{p-1}$ will be linearly independent in general (we are in $\mathbb{R}^{n-p}$). Finally, the point $x_p$ is determined from $x_1,\ \ldots,\ x_{p-1}$ by a (complicated) equation involving other operators, so it will be linearly independent of the previous ones in general.

We now turn to numerical computations.

**f. Numerical computations.**

We want to find $p$ such that $l,\ h$ exist, satisfying

$$\begin{cases} l \ \geq\ 2 \\ (h+1)(p-lh-1) \ \geq\ n-p \end{cases}$$

The last condition gives

$$\frac{ph-h+2p-1-n}{h(h+1)} \ \geq\ l.$$

which is equivalent to

$$-2h^2+(p-3)h+2p-1-n \ \geq\ 0. \tag{2.f.1}$$

The maximum is reached for $h=(p-3)/4$, which is not necessarily an integer. Thus we write

$$p=4r+\rho, \qquad \rho=0,1,2,3,$$

and take $h=r$. Thus (2.f.1) becomes :

$$2r^2+r\rho+5r+2\rho-1-n \ \geq\ 0,$$

which is satisfied as soon as :

$$r \ \geq\ \frac{-\rho}{4}-\frac{5}{4}+\frac{1}{4}\sqrt{\rho^2-6\rho+33+8n}\ ,$$

or

$$p \ \geq\ -5+\sqrt{\rho^2-6\rho+33+8n}\ ,$$

and this condition is satisfied, no matter what value $\rho$ takes $(\rho=0,1,2,3)$, if

$$p \ \geq\ \sqrt{8n+33}-5\ . \tag{2.f.2}$$

**g. A numerical example.**

We explain on an example how one passes from equations (2.e.1) to the long equations (2.e.5).

Take $n=101,\ p=\sqrt{8n+33}-5=\sqrt{841}-5=24,\ n-p=77$ is odd.

Thus there are $p=24$ equations (2.e.1), with $23$ variables.
Divide $p$ by $4$ : $24=4\times 6$, so $r=6$, and $h=6,\ l=2,\ q=h.l=12$.
Consider the last $12$ equations. Here $p-q-1=24-12-1=11$.
Using the last equation, compute $x_{23}$ in terms of $x_1,\ldots,x_{11}$ ; using the 23rd equation, compute $x_{22}$ in terms of $x_1,\ldots,x_{11}$, and so on ; using the 13th equation, compute $x_{12}$ in terms of $x_1,\ldots,x_{11}$.
Here $L_0=(x_1,\ldots,x_{11}),\ L_1=(x_{12},\ldots,x_{17}),\ L_2=(x_{18},\ldots,x_{23})$.
In order to build the first long equation, kill $L_2$, substitute $x_{12},\ldots,x_{17}$ using $x_1,\ldots,\ x_{11}$ : this gives $11+6\times 11=77$ variables.

In order to build the second long equation, kill $L_1$, substitute $x_{18},\ldots,x_{23}$ using $x_1,\ldots,x_{11}$ : this again gives $77$ variables.

We now turn to the general assignment problem : not all numbers are real. We use the distinction “essentially real case”, “essentially complex case” established in Part I.

**3.– The essentially real case.**

**Theorem 3.1.** – *If $p \geq \sqrt{8n+33}-5$ $(n-p$ odd) or $p \geq \sqrt{8n+33}-4$ $(n-p$ even), the assignment of an essentially real set $\{\lambda_1, \dots, \lambda_n\}$ is possible in general.*

**Proof** of Theorem 3.1. – Here our subspaces $F_j = R(\lambda_j)F$ $(j=1,\dots,p)$ have the same meaning as before ; they are subspaces of $\mathbb{R}^n$, and are described by (2.a.1).

The $N_j = (A-\lambda_j I)N$ are real for $j=p+1,\dots,n-2k$ and are described by a real operator $V_j$ : $\mathbb{R}^{n-p} \to \mathbb{R}^p$, by (2.a.2). For $j=n-2k+1,\dots,n-k$, they are complex and $V_j$ : $\mathbb{C}^{n-p} \to \mathbb{C}^p$.

The vectors $u_j$ are in $\mathbb{R}^n$, $j=1,\dots,p$.

Equations (2.b.1) (2.b.2) now become :

$$\begin{cases} T_1(a_{1,1}u_1' + \cdots + a_{1,p}u_p') &= a_{1,1}u_1'' + \cdots + a_{1,p}u_p'' \\ & \vdots \\ T_p(a_{p,1}u_1' + \cdots + a_{p,p}u_p') &= a_{p,1}u_1'' + \cdots + a_{p,p}u_p'' \end{cases} \tag{3.1}$$

$$\begin{cases} a_{p+1,1}u_1' + \cdots + a_{p+1,p}u_p' &= V_{p+1}(a_{p+1,1}u_1'' + \cdots + a_{p+1,p}u_p'') \\ & \vdots \\ a_{n-2k,1}u_1' + \cdots + a_{n-2k,p}u_p' &= V_{n-2k+1}(a_{n-2k,1}u_1'' + \cdots + a_{n-2k,p}u_p'') \end{cases} \tag{3.2}$$

$$\begin{cases} a_{n-2k+1,1}u_1' + \cdots + a_{n-2k+1,p}u_p' &= V_{n-2k+1}(a_{n-2k+1,1}u_1'' + \cdots + a_{n-2k+1,p}u_p'') \\ & \vdots \\ a_{n-k,1}u_1' + \cdots + a_{n-k,p}u_p' &= V_{n-k}(a_{n-k,1}u_1'' + \cdots + a_{n-k,p}u_p'') \end{cases} \tag{3.3}$$

If $p \leq n/3$ (the interesting case !), $n-2k-p \geq p$, and equations (3.2) can be used to compute $u_1'$ in terms of $u_1'', \dots, u_p'$ in terms of $u_p''$, as we did in (2.c.1). But in all cases, we can argue as follows :

**Proposition 3.2.** – *For any choices of $u_1, \dots, u_{p-1}$, all non-zero, there is a choice of $u_p$, non-zero, and for each $i=p+1$, a choice of scalars $a_{i,1}, \dots, a_{i,p}$ (real if $i \leq n-2k$, complex if $i > n-2k$), at least one non-zero in each list, such that equations (3.2) and (3.3) are simultaneously satisfied.*

**Proof**. – We consider the function $\varphi$, defined on the unit sphere of $\mathbb{R}^n$, with values in $\mathbb{R}^{n-p}$ :

$$\begin{aligned} \varphi(V) \;=\; \big(& det(u_1' - V_{p+1}u_1'', \dots, u_{p-1}' - V_{p+1}u_{p-1}'', v' - V_{p+1}v''), \\ & \vdots \\ & det(u_1' - V_{n-2k}u_1'', \dots, u_{p-1}' - V_{n-2k}u_{p-1}'', v' - V_{n-2k}v''), \\ & Re\ det(u_1' - V_{n-2k+1}u_1'', \dots, u_{p-1}' - V_{n-2k+1}u_{p-1}'', v' - V_{n-2k+1}v''), \\ & Im\ det(u_1' - V_{n-2k+1}u_1'', \dots, u_{p-1}' - V_{n-2k+1}u_{p-1}'', v' - V_{n-2k+1}v''), \\ & \vdots \\ & Re\ det(u_1' - V_{n-k}u_1'', \dots, u_{p-1}' - V_{n-k}u_{p-1}'', v' - V_{n-k}v''), \\ & Im\ det(u_1' - V_{n-k}u_1'', \dots, u_{p-1}' - V_{n-k}u_{p-1}'', v' - V_{n-k}v'')\big), \end{aligned}$$

where $v = (v', v'')$, $v' \in \mathbb{R}^p$, $v'' \in \mathbb{R}^{n-p}$.

This is an odd function of $v$ ; Borsuk's Theorem applies and solves all equations (3.2), (3.3) at the same time. The $v$ obtained is non-zero.

Now, we are left with equations (3.1), in which we take all $a_{i,p}=0$, $i=1,\dots,p$, so as to kill $u_p$.

We now decide to fix arbitrary operators $S_1,\dots,S_{p-1}, \mathbb{R}^{n-p} \to \mathbb{R}^p$, such that all $T_iS_j-I$ are invertible and independent operators. We then set :

$$u'_1 \;=\; S_1u''_1,\dots,u'_{p-1} \;=\; S_{p-1}u''_{p-1}. \tag{3.4}$$

This way, equations (3.1) become, with $U_{i,j}=T_iS_j-I$ and $x_j=u''_j$ :

$$\begin{cases} a_{1,1}U_{1,1}x_1+\cdots+a_{1,p-1}U_{1,p-1}x_{p-1} &= 0\\ & \vdots\\ a_{p,1}U_{p,1}x_1+\cdots+a_{p,p-1}U_{p,p-1}x_{p-1} &= 0\end{cases} \tag{3.5}$$

which is (2.e.1) and is solved in the same manner.

**4. The essentially complex case**.

**Theorem 4.1.** – *The assignment of an essentially complex set* $\{\lambda_1,\dots,\lambda_n\}$ *is possible in general as soon as :*

$$\begin{aligned} p &\ge 2\sqrt{8n+89}-18, && n \text{ odd}, \ p \text{ even},\\ p &\ge 2\sqrt{8n+89}-17, && n \text{ odd}, p \text{ odd},\\ p &\ge 2\sqrt{12n+181}-26, && n \text{ even}, p \text{ even},\\ p &\ge 2\sqrt{12n+181}-25, && n \text{ even}, p \text{ odd}.\end{aligned}$$

**Proof** of Theorem 4.1. – Decreasing $p$ by one unit if necessary, we assume $p$ to be even and set $\mu=p/2$.

Now, we have only $\mu$ $F_j$'s, and they are complex. We try to reproduce the notation of the real case, by putting :

$$F_1 \;=\; R(\lambda_{n-k-\mu+1})F,\dots,F_\mu \;=\; R(\lambda_{n-k})F. \tag{4.1}$$

These subspaces are $p$-dimensional in $\mathbb{C}^n$.

By (2.a.1), we write them as :

$$F_j \;=\; \{z\in\mathbb{C}^n \;;\; z'' \;=\; T_jz'\}, \qquad j=1,\dots,\mu, \tag{4.2}$$

where each $T_j\ :\mathbb{C}^p\to\mathbb{C}^{n-p}$.

The real subspaces $(A-\lambda_1I)N,\dots,(A-\lambda_{n-2k}I)N$, all of dimension $n-p$, are written :

$$N_{\mu+1} \;=\; (A-\lambda_1I)N,\dots,N_{\mu+n-2k} \;=\; (A-\lambda_{n-2k}I)N, \tag{4.3}$$

with

$$N_j \;=\; \{z\in\mathbb{R}^n \;;\; z' \;=\; V_jz''\}, \qquad j=\mu+1,\dots,\mu+n-2k, \tag{4.4}$$

and $V_j\ :\mathbb{R}^{n-p}\to\mathbb{R}^p$.

(Note that these $N_j$'s will not exist at all if $n=2k$ : case when the assignment is entirely complex.)

Finally, the complex subspaces $(A-\lambda_{n-2k+1}I)N,\dots,(A-\lambda_{n-k-\mu}I)N$ are written as

$$N_{\mu+n-2k+1} \;=\; (A-\lambda_{n-2k+1}I)N,\dots,N_{n-k} \;=\; (A-\lambda_{n-k-\mu}I)N, \tag{4.5}$$

and

$$N_j \;=\; \{z\in\mathbb{C}^n \;;\; z' \;=\; V_jz''\}, \qquad j=\mu+n-2k+1,\dots,n-k, \tag{4.6}$$

and $V_j\ :\mathbb{C}^{n-p}\to\mathbb{C}^p$.

In order to simplify the notation, we set $\nu=\mu+n-2k$.

The complex subspace $H$ will be determined by $\mu$ independent vectors $u_1, \ldots, u_\mu$ in $\mathbb{C}^n$. The fact that $H$ intersects $F_1, \ldots, F_\mu$ can be expressed by :

$$\left\{ \begin{array}{ll} T_1(a_{1,1}u_1' + \cdots + a_{1,\mu}u_\mu') & = a_{1,1}u_1'' + \cdots + a_{1,\mu}u_\mu'' \\ & \vdots \\ T_\mu(a_{\mu,1}u_1' + \cdots + a_{\mu,\mu}u_\mu') & = a_{\mu,1}u_1'' + \cdots + a_{\mu,\mu}u_\mu'' \end{array} \right. \tag{4.7}$$

where the $a_{i,j}$ are real, one at least on each line being non-zero.

The fact that $ReH$ intersects $N_{\mu+1}, \ldots, N_\nu$ is expressed by :

$$\left\{ \begin{array}{l} a_{\mu+1,1}(u_1' + \overline{u}_1') + \cdots + a_{\mu+1,\mu}(u_\mu' + \overline{u}_\mu') \ = \\ \qquad\qquad = \ V_{\mu+1}\big(a_{\mu+1,1}(u_1'' + \overline{u}_1'') + \cdots + a_{\mu+1,\mu}(u_\mu'' + \overline{u}_\mu'')\big) \\ \qquad\qquad\qquad \vdots \\ a_{\nu,1}(u_1' + \overline{u}_1') + \cdots + a_{\nu,\mu}(u_\mu' + \overline{u}_\mu') \ = \\ \qquad\qquad = \ V_{\nu,1}\big(a_{\nu,1}(u_1'' + \overline{u}_1'') + \cdots + a_{\nu,\mu}(u_\mu'' + \overline{u}_\mu'')\big) \end{array} \right. \tag{4.8}$$

where the $a_{i,j}$ are also real, and finally the fact that $H \oplus \overline{H}$ intersects $N_{\nu+1}, \ldots, N_{n-k}$ is translated into :

$$\left\{ \begin{array}{l} a_{\nu+1,1}u_1' + b_{\nu+1,1}\overline{u}_1' + \cdots + a_{\nu+1,\mu}u_\mu' + b_{\nu+1,\mu}\overline{u}_\mu' \ = \\ \qquad\qquad = \ V_{\nu+1}(a_{\nu+1,1}u_1'' + b_{\nu+1,1}\overline{u}_1'' + \cdots + a_{\nu+1,\mu}u_\mu'' + b_{\nu+1,\nu}\overline{u}_\mu'') \\ \qquad\qquad\qquad \vdots \\ a_{n-k,1}u_1' + b_{n-k,1}\overline{u}_1' + \cdots + a_{n-k,\mu}u_\mu' + b_{n_k,\mu}\overline{u}_\mu' \ = \\ \qquad\qquad = \ V_{n-k}(a_{n-k,1}u_1'' + b_{n-k,1}\overline{u}_1'' + \cdots + a_{n-k,\mu}u_\mu'' + b_{n-k,\nu}\overline{u}_\mu'') \end{array} \right. \tag{4.9}$$

In these equations, the $a_{i,j}$ and $b_{i,j}$ are complex numbers, and on each line at least one of the $a$'s or one of the $b$'s should be non-zero.

We first consider the equations (4.9).

**Proposition 4.2.** – *For every choice of $u_1, \ldots, u_{\mu-1}$, non-zero, there is a non-zero $u_\mu$ and scalars $a_{i,j}$, $b_{i,j}$ (at least one non-zero on each line) such that equations (4.9) are satisfied.*

**Proof** of Proposition 4.2. – We observe that $span_{\mathbb{C}^n}\{z, \overline{z}\} = span_{\mathbb{C}^n}\{Re\ z, Im\ z\}$, for any $z \in \mathbb{C}^n$.

So if we set $v_j = Re\ u_j$, $w_j = Im\ u_j \in \mathbb{R}^n$, equations (4.9) will be satisfied if we find complex scalars $a_{i,j}'$, $b_{i,j}'$ (at least one non-zero on each line), such that :

$$\left\{ \begin{array}{l} a_{\nu+1,1}'(v_1' - V_{\nu+1}v_1'') + b_{\nu+1,1}'(w_1' - V_{\nu+1}w_1'') + \cdots + \\ \qquad\qquad + a_{\nu+1,\mu}'(v_\mu' - V_{\nu+1}v_\mu'') + b_{\nu+1,\mu}'(w_\mu' - V_{\nu+1}w_\mu'') \ = \ 0 \\ \qquad\qquad\qquad \vdots \\ a_{n-k,1}'(v_1' - V_{n-k}v_1'') + b_{n-k,1}'(w_1' - V_{n-k}w_1'') + \cdots + \\ \qquad\qquad + a_{n-k,\mu}'(v_\mu' - V_{n-k}v_\mu'') + b_{n-k,\mu}'(w_\mu' - V_{n-k}w_\mu'') \ = \ 0 \end{array} \right. \tag{4.10}$$

We now consider a function $\varphi$, defined on the unit ball of $\mathbb{R}^n$, by :

$$\varphi(w) \ = $$

$$\begin{aligned}
\big(&Re\ det(v_1' - V_{\nu+1}v_1'', w_1' - V_{\nu+1}w_1'', \ldots, v_{\mu-1}' - V_{\nu+1}v_{\mu-1}'',\\
&\qquad\qquad\qquad\qquad w_{\mu-1}' - V_{\nu+1}w_{\mu-1}'', v_\mu' - V_{\nu+1}v_\mu'', w' - V_\nu w''),\\
&Im\ det(v_1' - V_{\nu+1}v_1'', w_1' - V_{\nu+1}w_1'', \ldots, v_{\mu-1}' - V_{\nu+1}v_{\mu-1}'',\\
&\qquad\qquad\qquad\qquad w_{\mu-1}' - V_{\nu+1}w_{\mu-1}'', v_\mu' - V_{\nu+1}v_\mu'', w' - V_\nu w''),\\
&\vdots\\
&Re\ det(v_1' - V_{n-k}v_1'', w_1' - V_{n-k}w_1'', \ldots, v_\mu' - V_{n-k}v_\mu'', w - V_{n-k}w''),\\
&Im\ det(v_1' - V_{n-k}v_1'', w_1' - V_{n-k}w_1'', \ldots, v_\mu' - V_{n-k}v_\mu'', w - V_{n-k}w'')\big)
\end{aligned}$$

Each determinant is well-defined, since it consists of $2\mu = p$ vectors, each in $\mathbb{C}^p$. The function $\varphi$ takes its values in $\mathbb{R}^{2k-p}$, and $2k - p \leq n - p < n$, and it is odd. So Borsuk's theorem applies, and gives a $w$, $\|w\| = 1$, such that all determinants are $0$. This proves Proposition 4.2 ; the real part of $u_\mu$, that is $v_\mu$, can be chosen arbitrarily.

We now turn to equations (4.8), which may not exist at all. We use the existing ones to compute some of the $u_j'$ in terms of the $u_j''$, in a trivial manner :

$$u_1' \ = \ V_{\mu+1}u_1'' \ , \ldots, \ u_{n-2k}' \ = \ V_\nu u_{n-2k}'' \tag{4.11}$$

If their number is unsufficient $(n-2k < \mu-1)$, we complete (4.11) in an arbitrary manner (as we did in § 3), with arbitrary linear operators $S_{n-2k+1}, \ldots, S_{\mu-1} \ : \mathbb{C}^{n-p} \to \mathbb{C}^p$, so as to get (with a change of notation) :

$$u_1' \ = \ S_1 u_1'' \ , \ \ldots, \ u_{\mu-1}' \ = \ S_{\mu-1}u_{\mu-1}'' \tag{4.12}$$

We now turn to (4.7). We eliminate $u_\mu$, substitute the $u_j'$ using (4.12), and get

$$\left\{\begin{aligned}
a_{1,1}(T_1S_1 - I)u_1'' + \cdots + a_{1,\mu-1}(T_1S_{\mu-1} - I)u_{\mu-1}'' \ &= 0\\
&\vdots\\
a_{\mu,1}(T_\mu S_1 - I)u_1'' + \cdots + a_{\mu,\mu-1}(T_\mu S_{\mu-1} - I)u_{\mu-1}'' \ &= 0
\end{aligned}\right. \tag{4.13}$$

We set $x_j = u_j'' \in \mathbb{C}^{n-p}$, $U_{i,j} = T_iS_j - I$, and obtain :

$$\left\{\begin{aligned}
a_{1,1}U_{1,1}\ x_1 + \cdots + a_{1,\mu-1}U_{1,\mu-1}\ x_{\mu-1} \ &= 0\\
&\vdots\\
a_{\mu,1}U_{\mu,1}\ x_1 + \cdots + a_{\mu,\mu-1}U_{\mu,\mu-1}\ x_{\mu-1} \ &= 0
\end{aligned}\right. \tag{4.14}$$

The $U_{i,j}$ are fixed operators $\mathbb{C}^{n-p} \to \mathbb{C}^{n-p}$, the $x_i$'s are at our disposal and should be non-zero, the $a_{i,j}$'s are complex numbers at our disposal and one at least one each line should be non-zero.

We are now in a situation identical to the one of the previous paragraphs, with $p$ replaced by $\mu$.

If $n$ is odd, $n-p$ is also odd, and we follow the same substitution procedure as in the previous paragraphs.

We let $q = l.h$ ; in the last $q$ equations we find $x_{\mu-q}, \ldots, x_{\mu-1}$ in terms of $x_1, \ldots, x_{\mu-q-1}$. Substituting in the first $l$ equations gives them a length of $(h+1)(\mu-q-1)$, and we request

$$(h+1)(\mu - q - 1) \ \geq \ n - 2\mu. \tag{cond 3}$$

The number of middle equations is $\mu - q - l$, and the number of variables at our disposal is $\mu - q - 2$, so we request also

$$l \ \geq \ 2. \tag{$cond$ 4}$$

After these substitutions have been performed, our first $l$ equations are :

$$\begin{cases} t_{1,1}A_{1,1}x_1 + \cdots + t_{1,n-p}A_{1,n-p}x_1 & = 0 \\ & \vdots \\ t_{l,1}A_{l,1}x_1 + \cdots + t_{l,n-p}A_{l,n-p}x_1 & = 0 \end{cases} \tag{4.15}$$

We consider

$$\begin{aligned} \varphi(x) \ = \ \big( & Re\ det(A_{1,1}x, \dots, A_{1,n-p}x), \\ & Im\ det(A_{1,1}x, \dots, A_{1,n-p}x), \\ & \vdots \\ & Re\ det(A_{l,1}x, \dots, A_{l,n-p}x), \\ & Im\ det(A_{l,1}x, \dots, A_{l,n-p}x)\big), \end{aligned}$$

which is an odd function since $n-p$ is odd, of $x \in \mathbb{C}^{n-p} = \mathbb{R}^{2n-2p}$, $\|x\| = 1$, taking its values in $\mathbb{R}^{2l}$, $l < \mu < p$. There is an $x_1$ such that $\|x_1\| = 1$ and $\varphi(x_1) = 0$, and this solves (4.15).

Now for the conditions (cond 3) and (cond 4) the same numerical computations as in § 2 show that they are satisfied as soon as :

$$\mu \ \geq \ \sqrt{8n+89} - 9,$$

that is

$$p \ \geq \ 2\sqrt{8n+89} - 18.$$

If $p$ was odd, then

$$p \ \geq \ 2\sqrt{8n+89} - 17.$$

Now, if $n$ is even, we cannot decrease $p$ by one unit, since we assumed it to be even. So we have to modify slightly the substitution procedure, sparing one variable (say $x_1$). This is done as follows.

We let $q = l.h$ as before, and in the last $q$ equations we find $x_{\mu-q}, \dots, x_{\mu-1}$ in terms of $x_2, \dots, x_{\mu-q-1}$. Substituting in the first $l$ equations gives them a length of $1 + (h+1)(\mu - q - 2)$, and we request

$$1 + (\mu - q - 2)(h+1) \ \geq \ n - 2\mu. \tag{$cond$ 5}$$

The number of middle equations is $\mu - q - l$ and the number of variables at our disposal is now only $\mu - q - 3$, so we request also

$$l \ \geq \ 3. \tag{$cond$ 6}$$

After these substitutions have been performed our first $l$ equations are :

$$\begin{cases} t_{1,1}A_{1,1}x_1 + t_{1,2}A_{1,2}x_2 + \cdots + t_{1,n-p}A_{1,n-p}x_2 & = 0 \\ & \vdots \\ t_{l,1}A_{l,1}x_1 + t_{l,2}A_{l,2}x_2 + \cdots + t_{l,n-p}A_{l,n-p}x_2 & = 0 \end{cases} \tag{4.16}$$

We consider

$$\begin{aligned}\varphi(x) \;=\; \big(&Re\ det(A_{1,1}x, A_{1,2}x_2, \ldots, A_{1,n-p}x_2),\\ &Im\ det(A_{1,1}x, A_{1,2}x_2, \ldots, A_{1,n-p}x_2),\\ &\vdots\\ &Re\ det(A_{l,1}x, A_{l,2}x_2, \ldots, A_{l,n-p}x_2),\\ &Im\ det(A_{l,1}x, A_{l,2}x_2, \ldots, A_{l,n-p}x_2)\big),\end{aligned}$$

where $x \in \mathbb{C}^{n-p} = \mathbb{R}^{2n-2p}$, $\|x\| = 1$, and $\varphi$ is odd and takes its values in $\mathbb{R}^{2l}$, $l < \mu < p$. For any fixed $x_2$, $\|x_2\| = 1$, there is, by Borsuk's Theorem, an $x_1$ such that $\|x_1\| = 1$, and $\varphi(x_1) = 0$, which solves (4.16).

Now, for the conditions (cond 5) and (cond 6), the same numerical computations as in § 2 show that they are satisfied as soon as :

$$\begin{aligned}\mu &\geq \sqrt{12n+181} - 13,\\ p &\geq 2\sqrt{12n+181} - 26,\end{aligned}$$

and if $p$ was odd, since we decreased $p$ by one unit at the beginning, we finally get the condition

$$p \geq 2\sqrt{12n+181} - 25,$$

under which the conclusion of Theorem 4.1 holds in all cases.


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