A measure-preserving map of the sphere onto a disk following Archimedes

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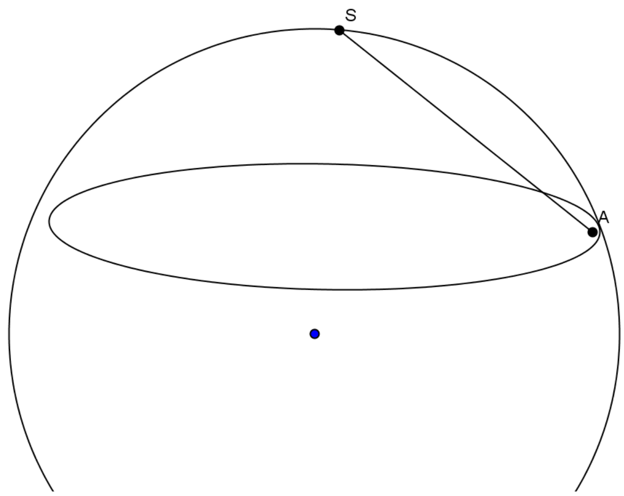
Introduction

To obtain plane representations of the Earth has been a long standing question, which, so far, has received only non-satisfactory answers. Of course, any projection will do locally, but all deteriorate in some sense globally. The most widely used one is the so-called Mercator projection : projection of the Earth onto a cylinder, tangent to the equator. Then, of course, the more North are the countries, the bigger they look on the projection.

Following Archimedes' work (see [1]), we build a transformation from the half-sphere to the disk which is measure-preserving : the size of the image of any country depends only on the size of the country, not on its location on the map.

In his work "On the sphere and on the cylinder" (see for instance our presentation [1]), Archimedes states the following theorem:

Theorem 1 (Archimedes). – *Let any spherical cup, with summit S and let A be any point of the boundary of this spherical cup. Then the cup has the same area as a disk of radius SA.*



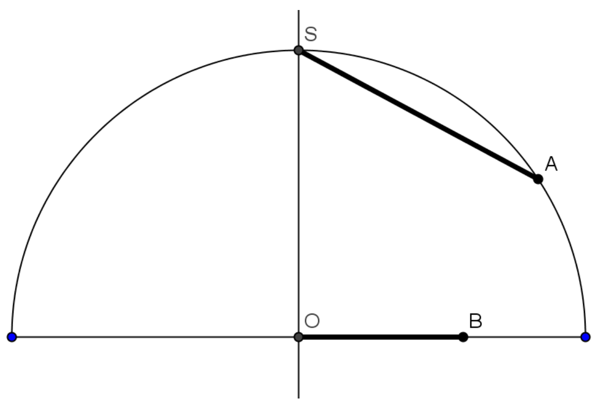
Let now HS be the half-sphere with summit S. Let D be the boundary disk, with center O.

From Archimedes' theorem, we deduce:

Theorem 2. – *Let A be any point of the half-sphere HS, and let B be the point of the disk D defined from A as follows :*

– *The point B is on the plane defined by OAS (think of it as a "vertical" plane, the disk being horizontal and the half-sphere lying on it), and on the same side of the vertical OS ;*

– *The distance OB satisfies $OB = \frac{1}{\sqrt{2}} SA$*



Then the application $f : A \to B$ is a measure preserving bijection from the half-sphere onto the disk.

Proof of Theorem 2

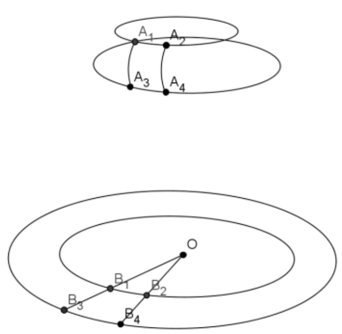
We may just as well consider that the radius of the sphere is 1. Quite clearly, the map f is injective : two different points A have different images. Clearly also, it is surjective : when A is on the boundary disk, then $SA = \sqrt{2}$ and $B = A$.

Let A_1, A_2 be two points with the same vertical coordinate (the vertical axis is the axis OS). They are transformed into two points B_1, B_2. Since the map operates vertically, the slice SA_1A_2 of the half-sphere is transformed into the angle OB_1B_2 of the disk, and identical slices will have identical projections, no matter where they are.

Now, take two points A_1, A_2 on the same vertical plane (points with same longitude).

By Theorem 1, the area of the cup limited by A_1 is $\pi(SA_1)^2$, that is twice the area of the circle of radius OB_1. The same way, the area of the cup limited by A_2 is twice the area of the circle of radius OB_2.

Therefore, the area of the "spherical band", that is the portion of sphere between both cups, is equal to twice the area of the annulus between both circles. This area does not depend on the precise location of the points A_1 and A_2.



Taking the intersection of a slice and of a spherical band, we see that this intersection is transformed into the intersection of an angle and of an annulus, and that the area of this intersection does not depend on the precise location of its original parts. On the picture left, the points A_1 and A_3 are on the same vertical plane ; they are transformed into B_1 and B_3 respectively ; the same for A_2 and A_4. The size of the elementary "rectangle" $B_1B_2B_3B_4$ depends only on the size of the elementary "rectangle" $A_1A_2A_3A_4$, and not on its location.

This proves our theorem.

We will call the application $f : A \to B$ defined above the "Archimedes map" of the half-sphere onto the disk.

Corollary. – *Take the Northern hemisphere as a half-sphere, with S = North Pole, then Archimedes map provides a measure-preserving transformation from the Northern hemisphere onto a disk, which preserves the relative importance of the territories.*

Remark : we call here "measure preserving" a map f which has the following property : if A_1 and A_2 have same measure, then $f(A_1)$ and $f(A_2)$ have the same measure. One does not necessarily have the same normalizations in the origin space and destination space of f. In our example, if one wants $m(f(A)) = m(A)$, one should consider the disk of radius $\sqrt{2}$ instead of the unit disk.

Reference

[1]. Bernard Beauzamy : Un théorème d'Archimède et sa démonstration d'origine. Séminaire exceptionnel tenu par la SCM le 5 mai 2010, à l'occasion du 2222ème anniversaire de la mort d'Archimède.

Map of the Northern Hemisphere, from Archimedes' map

(Courtesy François Guénard, University of Orsay, France ; drawn using Mathematica)