The error law on a measurement with unknown precision

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Summary

Consider a situation (very frequent in practice) where the value of a measurement is given, with no indication at all about the uncertainty: for instance, the Weather Bureau says that now the temperature is 5°C, but says nothing about the precision. We show how to construct a probability law for the uncertainty, in a way that requires only minimal information; it relies upon two numbers only: the expected worst precision and the expected best precision. This minimal law is of logarithmic type.

I. Description of the problem

Usually, when a physical measurement is made, an uncertainty is associated to it. It can be under one of the following three forms:

– An absolute interval : the measurement gave x_0 and we are certain that the true value is between $x_0 - a$ and $x_0 + a$ (here, this interval is given as symmetric, but this does not need to be the case in practice) ;

- A confidence interval: the measurement gave x_0 and we consider that the true value is between $x_0 - a$ and $x_0 + a$, with 95 % probability (meaning that, if the experiment is repeated a large number of times, then the true value will be in this interval 95 % of the times).

- A probability law: it is given by a density $f(t)$ and the probability that the true value is in any interval $[a,b]$ is $\int_a^b f(t)dt$.

The third form is of course much more satisfactory, but it is almost never provided. At best, forms 1 or 2 are provided, and from that people deduce some probability law, for instance uniform or triangle. This is usually totally arbitrary, and therefore should be avoided.

But in many cases, no interval at all is provided, or only an absolute interval which is so large that it cannot be used in practice.

A confusion is often made between the precision of the measurement and the extreme possible values. For instance, the temperature announced by Météo France at this time is 5°C, but they say nothing about the precision of the measurement. I can certainly assume that the true value is between -50°C and +50°C, because no value outside this range has ever been seen in Paris. This range is certainly too wide for applications.

When the indication given is 5°C, with no more information, we can certainly assume that the measurement was in the range $4.5-5.5\,°$C, but this gives no information at all upon the precision of the sensor itself. This precision may be anything between $0.1°C$ and $1°C$. So, the best precision we get here is $0.5+0.1$ (optimistic) and the worst is $0.5+1$ (pessimistic). We make our construction from these two numbers. Note that this "precision" includes all possible elements which deteriorate the measurement, including, as we see here, a rounding-off error in the transcription.

II. The results

We are going to show :

Theorem. - *Let b be the best estimated precision for the measurement and w the worst estimated precision. Then the Minimal Information Law for the error is the probability law defined by the following formulas :*

$$f(x) = \frac{Log(w) - Log(b)}{2(w-b)} \text{ (constant) if } x_0 - b \le x \le x_0 + b$$

$$f(x) = \frac{Log(w) - Log|x - x_0|}{2(w-b)} \text{ if } x_0 + b \le x \le x_0 + w \text{ or } x_0 - w \le x \le x_0 - b$$

$f(x)=0$ if $x \le x_0 - w$ or $x \ge x_0 + w$.

Before we prove the theorem, let us explain the idea behind the statement. Let us take a very simple case, where two values only are possible for the precision : say $b=1$ and $w=2$. Say the measurement was $x_0 = 0$. So, if the sensor has precision 1, we consider all points with identical probability on the interval $\left[-1,1\right]$: this corresponds to a uniform law on this interval. This law has density :

$$f_1(x)=\frac{1}{2}1_{[-1,1]}(x)$$

where the notation $1_A(x)$ is the characteristic function of the set A, that is the function which has value 1 if $x \in A$, 0 otherwise.

Similarly, if the precision is 2, then we have the uniform density :

$$f_2(x)=\frac{1}{4}1_{[-2,2]}(x)$$

Since we do not know if the precision is 1 or 2, we attribute probability $1/2$ to each, and our density is :

$$f=\frac{1}{2}\left(f_1+f_2\right)$$

This function has the following form :

$$f(x)=\frac{3}{8} \text{ if } -1 \le x \le 1$$

$$f(x)=\frac{1}{8} \text{ if } -2 \le x \le -1 \text{ or } 1 \le x \le 2$$

$$f(x)=0 \text{ if } x \le -2 \text{ or } x \ge 2.$$

More generally, assume that the precision may be any number between 1 and N. If the precision is n, we have a uniform law on the interval $\left[-n,n\right]$, that is :

$$f_n(x)=\frac{1}{2n}1_{[-n,n]}(x)$$

Each of these precisions has the same probability, namely $\frac{1}{N}$, and the final estimate is :

$$f(x) = \frac{1}{N}\sum_{n=1}^{N} f_n(x)$$

On a given interval $[k-1,k]$, only the functions f_n with $n \geq k$ bring a contribution, that is :

$$f(x) = \frac{1}{2N}\sum_{n=k}^{N}\frac{1}{n}$$

We may now prove the theorem in the general case.

Proof of the theorem.

The precision is between b and w, with of course $0 \leq b \leq w$. Assume that the precision is u, $b \leq u \leq w$. Then we have a uniform law on $[x_0 - u, x_0 + u]$, that is :

$$f_u(x) = \frac{1}{2u} 1_{[x_0-u, x_0+u]}(x)$$

None of these precisions is more likely than the others, so u should follow a uniform law on the interval $[b,w]$. Our final precision will be :

$$f(x) = \frac{1}{b-w}\int_b^w \frac{1}{2u} 1_{[x_0-u, x_0+u]}(x)\, du$$

Let us compute explicitly this function.

If $x_0 - b \leq x \leq x_0 + b$, all functions are non-zero, and :

$$f(x) = \frac{1}{b-w}\int_b^w \frac{du}{2u} = \frac{Log(w) - Log(b)}{2(w-b)}$$

If $x_0 + b \leq x \leq x_0 + w$, we have :

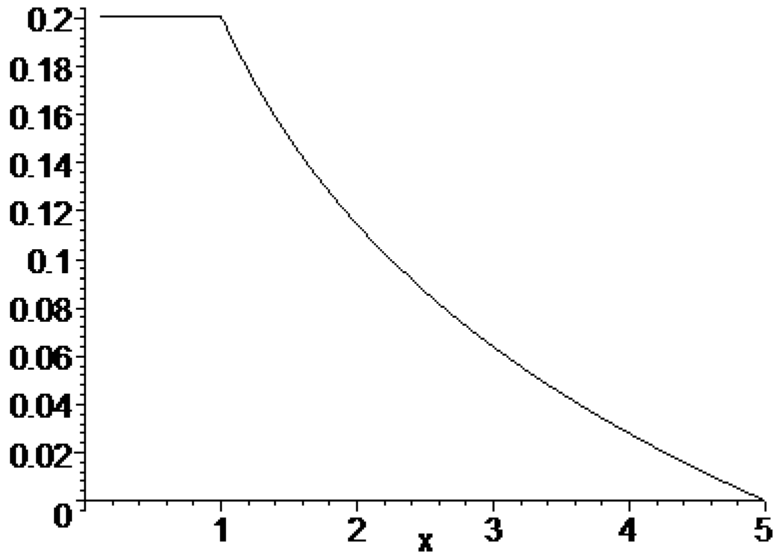
$$f(x) = \frac{1}{b-w}\int_{x-x_0}^{w} \frac{du}{2u} = \frac{Log(w) - Log(x-x_0)}{2(w-b)}$$

and the same if $x_0 - w \leq x \leq x_0 - b$.

Finally, if $x \geq x_0 + w$ or $x \leq x_0 - w$, then $f(x) = 0$.

This proves the theorem.

Here is an example of function f, for $b = 1$, $w = 5$, $x_0 = 0$:



Graph of the function f