Robust LP with Right-Handside Uncertainty, Duality and Applications

Michel Minoux*

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Abstract

The various robust linear programming models investigated so far in the literature essentially appear to be based either on what is referred to as 'rowwise' uncertainty models or on 'columnwise' uncertainty models (these basically assume that the rows - resp: the columns - of the constraint matrix are subject to changes within a well specified *uncertainty set*). In this paper, we discuss a special case of columnwise uncertainty namely the subclass of robust LP models with uncertainty limited to the right handside only. (this subclass does not appear to have been significantly investigated so far). In this context we introduce the concept of ' two-stage robust LP model' as opposed to the standard case (which might be referred to as 'single-stage robust LP model') and we address the question of whether LP duality can be used to convert a LP problem with RHS-uncertainty into a robust LP problem with uncertainty on the objective function. We show how to derive both statements of (a) the dual to the robust model and (b) the robust version of the dual. The resulting expressions of the objective function to be optimized in both cases, appear to be clearly distinct. Moreover, from a complexity point-of-view, one appears to be efficiently solvable (it reduces to a convex optimization problem) whereas the other, as a nonconvex optimization problem, is expected to be computationally difficult in the general case. As an application of the 2-stage robust LP model introduced here, we next investigate the *robust PERT scheduling problem*, considering two possible natural ways of specifying the uncertainty set for the task durations: the case where the uncertainty set is a scaled ball with respect to the L_∞ norm; the case where the uncertainty set is a scaled Hamming ball of bounded radius (which, though leading to a quite different model, bears some resemblance to the well-known Bertsimas-Sim approach to robustness). We show that in both cases, the resulting robust optimization problem can be efficiently solved in polynomial time.

1 Introduction

Various models for handling robustness objectives with respect to uncertainties on some specified coefficients in linear programming models have been proposed in the literature. We can mention Soyster (1973), Ben-Tal and Ne-

*University Paris-6, France - Email : Michel.Minoux@lip6.fr

mirovski (1998, 2000), Bertsimas and Sim (2003, 2004).

The various approaches proposed can roughly be divided into two distinct categories, depending on whether the underlying uncertainty model refers to possible fluctuations on the row vectors of the constraint matrix (we call this *'rowwise uncertainty'*), or on column vectors (we call this *'columnwise uncertainty'*).

Columnwise uncertainty was first considered by Soyster (1973). In this model each column A_j of the $m \times n$ constraint matrix is either supposed to be exactly known, or is only known to belong to a given subset $K_j \subset \mathbb{R}^m$ ('uncertainty set'). The cost vector and the right hand-side are supposed to be certain. A robust solution is a solution which is feasible for all possible choices of the uncertain column vectors in their respective uncertainty sets. With this definition, assuming nonnegativity constraints on all variables of the LP, it can easily be shown that the problem of finding an optimal robust solution reduces to solving an ordinary LP with constraint matrix $A = (a_{i,j})$ where, $\forall i, j$, the coefficient $a_{i,j}$ is defined as:

- $a_{i,j} = \max_{v \in K_j}\{v_i\}$ in case of a ith constraint of the form $\leq$
- $a_{i,j} = \min_{v \in K_j}\{v_i\}$ in case of a ith constraint of the form $\geq$

Note that the above maximization (or minimization) can easily be carried out if we assume the uncertainty sets either of finite cardinality (and not too big !), or closed convex.

As observed by many authors, a drawback of Soyster's model is that it usually leads to rather conservative solutions, in other words the price to pay for robustness in the above sense is often too high.

Contrasting with the above, rowwise uncertainty has attracted more interest and has been studied, among others, by Ben-Tal and Nemirovski (1998, 2000), and more recently by Bertsimas and Sim (2003, 2004).

Ben-Tal and Nemirovski start with the assumption that each row A_i of the constraint matrix belongs to a known uncertainty set consisting of an ellipsoid $E_i \subset \mathbb{R}^n$, and a solution $x \in \mathbb{R}^n, x \geq 0$ is said to be robust in this context iff it satisfies:

$$\text{for all } i : A_i x \leq b_i, \forall A_i \in E_i.$$

Ben-Tal and Nemirovski then show that finding an optimal robust solution reduces to solving a conic quadratic problem, which can be done in polynomial time.

A way to obviate nonlinearity, while retaining the idea of rowwise uncertainty, was proposed by Bertsimas and Sim (2003, 2004), considering a slightly different model of uncertainty. More precisely, they assume that each uncertain coefficient $a_{i,j}$ can take values in a given interval $[a_{i,j} - \alpha_{i,j}, a_{i,j} + \alpha_{i,j}]$ and, for each row i, a positive parameter $\Gamma_i > 0$ (not larger than the total number of uncertain coefficients in row i) is considered. A solution x is then qualified as $\Gamma-$robust (in the sense of Bertsimas and Sim) iff for all $i = 1, ..., m$ this solution satisfies the ith constraint for all possible choices of the coefficients in row i such that at most Γ_i of the uncertain coefficients in the row are allowed to deviate from the nominal values $a_{i,j}$. (note that the above statement implicitly assumes the Γ_i parameters to be integers, but Bertsimas and Sim show that a slightly more general definition, allowing for nonintegral values of the Γ_i's can be handled in the same way). With this model of uncertainty, Bertsimas and Sim show that finding an optimal $\Gamma-$robust solution can be reduced to solving an ordinary linear program only moderately increased in size, thus opening the way to large scale applications. Moreover the approach readily extends to optimization problems including integrality constraints on all or part of the variables, in that case the robust version of the problem is a MIP, but, again, the resulting robust model is only moderately increased in size as compared with the original model.

In the present paper we investigate a specific subclass of robust LP decision problems with columnwise uncertainty, namely LP problems with uncertainty on the right-handside coefficients only. To handle such problems, a first natural idea would be to use duality to reformulate them as robust LP's with uncertainty on the objective. This is the subject of section 2 below.

2 Duality and robustness for LP's with uncertainty in the RHS

We first address in this section the question of whether duality can prove in any way useful to convert a columnwise uncertain linear program into a rowwise uncertain linear program, assuming of course the same uncertainty model for the columns of the given linear program and for the corresponding rows in the dual.

Intuitively, no nice (i.e. strong) duality result is to be expected when taking into account robustness constraints since, in both the primal and the dual, there is a price to pay for uncertainty, therefore: if we maximize in the primal, the robust primal optimal solution value will be (in general strictly) less than the optimal solution value of the 'nominal' primal LP; and minimizing in the dual will lead to a robust dual optimal solution value (in general) strictly larger than the same value.

Let us illustrate the phenomenon on a small typical example. Consider the following LP (a continuous knapsack problem actually) with two uncertain coefficients a_1 and a_2 in the constraint matrix:

$$
(P)\quad \begin{array}{ll} \text{Maximize} & 4x_1 + 3x_2 \\ \text{s.t:} & \\ & a_1 x_1 + a_2 x_2 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}
$$

the standard LP dual of which reads:

$$
(D)\quad \begin{array}{ll} \text{Minimize} & 4u \\ \text{s.t:} & \\ & a_1 u \geq 4 \\ & a_2 u \geq 3 \\ & u \geq 0 \end{array}
$$

Let us assume that the uncertainty set for a_1 is the real interval $[2, 3]$, the uncertainty set for a_2 is the real interval $[1, 2]$, and let us take as definition of a robust solution in both (P) and (D) a solution which is feasible for any possible values of a_1 and a_2 in their respective uncertainty sets. Then it is easily seen that the optimal robust primal solution is $x^0 = [0, 2]$ with corresponding primal objective function value 6; and the optimal robust dual solution is $u^0 = 3$ with corresponding dual objective function value 12. This example thus clearly shows that no natural extension of the usual properties related to LP duality is to be expected in the context of robust Linear Programming.

A special case of columnwise uncertainty in Linear Programming is when uncertainty only concerns the coefficients of the right handside (RHS). Such problems frequently arise in practical applications. As a typical example, we mention the robust PERT scheduling problem with uncertainty on the durations of (some of) the tasks, assuming that a robust earliest termination date has to be determined. More precisely, we want to determine the minimum total duration of the project under any possible assignment of task durations, taken in a given uncertainty set. The 2-stage robust model discussed in 3.2 below will appear to be relevant to such applications.

Consider the following LP:

$$\begin{aligned} \max \quad & c^T x \\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

and assume that the right handside b is not known exactly, but only known to belong to some uncertainty set $B \subset \mathbb{R}^m$. The set B may be finite or infinite (we will introduce additional assumptions on this set when necessary).

Two distinct robustness models for LP's with uncertain RHS will be successively discussed in the following sections, namely single-stage robust decision models (Section 3) and two-stage robust decision models (Section 4). For both cases it will be shown that, even in the restricted situation addressed here (uncertainty in the RHS only) one cannot use standard duality theory to convert a columnwise uncertain linear program into a rowwise uncertain linear program while preserving equivalence. Also, examples will be provided to show that the 2-stage robust LP decision model is capable of producing less conservative solutions as compared with the single stage robust LP model.

3 Single-stage robust (LP) decision model

We first consider the simplest case where the values of all the decision variables x have to be fixed (taking into account uncertainty) before we get any kind of information on the actual realization of the uncertain parameters. In such a simple model (indeed a special case of Soyster's model) feasibility has to be ensured for any $b \in B$, and the problem to be solved simply reduces to:

$$\begin{aligned} \max \quad & c^T x \\ & Ax \le \underline{b} \\ & x \ge 0 \end{aligned}$$

where, $\forall i, \underline{b}_i = \min_{b \in B}\{b_i\}$

The (standard) LP dual to the above problem reads:

$$\begin{aligned} \text{(D1)} \quad \min \quad & u^T \underline{b} \\ & u^T A \ge c \\ & u \ge 0 \end{aligned}$$

On the other hand, if we consider the dual to:

$$\begin{aligned} \max \quad & c^T x \\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

we get:

$$\begin{aligned} \min \quad & u^T b \\ & u^T A \ge c \\ & u \ge 0 \end{aligned}$$

Now consider the robust version of this dual problem where the cost vector b is uncertain and can take any value in B. A simple and natural objective in this context is to find u achieving a minimum value of $\max u^T b$ over all possible $b \in B$, thus leading to:

$$\begin{aligned} \text{(D2)} \quad \min_u \max_{b \in B} \quad & \{u^T b\} \\ \text{s.t:} \quad & u^T A \ge c \\ & u \ge 0 \end{aligned}$$

It is clearly realized that (D1) and (D2) are completely different optimization problems on the same solution sets since,

$$\forall u \ge 0, u^T \underline{b} \le \max_{b \in B}\{u^T b\}$$

with strict inequality holding in the general case.

If the set B is either finite, or closed convex, it is observed that (D2) can be efficiently solved via standard convex optimization techniques since the function:

$$u \to \max_{b \in B}\{u^T b\}$$

is convex. Also, in this case, $\underline{b}$ can be efficiently computed since $\min_{b \in B}\{b_i\}$ too is a convex optimization problem, therefore problem (D1), too, can be efficiently solved.

4 Two-stage robust (LP) decision model

It frequently arises in applications that the process of decision-making under uncertainty can be decomposed in two successive steps (two-stage decision making) or more (multi-stage decision making). For simplicity of presentation, we restrict here to the case of two-stage decision making. In this case, the set of decision variables x is decomposed (partitioned) into two distinct sets of variables which we denote y and z. The y variables concern the decisions to be taken in the first stage (before knowing anything about which realization of uncertainty will arise) and the z variables concern the decisions to be taken in the second stage (after realization of uncertainty).

Limiting ourselves, to make the discussion easier to follow, to the case where the objective function only depends on the decision variables of the first stage, our decision problem can thus be rewritten:

$$\begin{array}{lll} \text{(I)} & \max & \gamma^T y \\ & \text{s.t:} & Fy + Gz \geq b \\ & & y \geq 0, z \geq 0 \end{array}$$

where γ, b are vectors and F and G are matrices of appropriate dimensions. (Observe that the reason for restricting to the case of an objective not depending on the z variables is only for the sake of simplicity in the presentation, our two-stage robust decision model would readily handle the general case of an objective depending on both the y and z variables).

Now, since the RHS b in (I) is uncertain, we have to make our robustness objectives precise. In the sequel, we consider robustness for a solution y by requiring that feasibility can be ensured for any possible RHS $b \in B$ by using the second stage decision variables z (by analogy with the terminology used in stochastic programming, the z variables might be referred to as 'recourse' variables). So, if we define $Y = \{y/y \geq 0 \text{ and } \forall b \in B, \exists z \geq 0 : Gz \leq b - Fy\}$, we want to solve:

$$\max_{y \in Y}\{\gamma^T y\}.$$

Note that in the above, for any given robust solution y, the value taken by the z variables depends on which $b \in B$ is actually realized. This is an important feature of our model which explains why it can produce less conservative solutions as compared with Soyster's model (see example given in Remark 1 below).

According to Farkas' Lemma, we know that, for fixed $b \in B$, a necessary and sufficient condition for the existence of $z \geq 0$ verifying $Gz \leq b - Fy$ is that $u^T(b - Fy) \geq 0$ for all u in the polyhedral cone:

$$C = \{u/u^T G \geq 0 \text{ and } u \geq 0\}.$$

Denoting $u^1, u^2, ..., u^p$, the extreme rays of the above cone, the set Y can equivalently be represented as the system of linear inequalities:

$$(u^j)^T Fy \leq (u^j)^T b, \forall b \in B, \forall j = 1, ...p$$

which is equivalent to:

$$(u^j)^T Fy \leq w_j = \min_{b \in B}(u^j)^T b.$$

The robust 2-stage decision problem is then reformulated as:

$$\text{(I)'} \quad \max \quad \gamma^T y$$
$$\text{s.t:} \quad (u^j)^T F y \leq w^j, \forall j = 1, ..., p$$
$$y \geq 0$$

Since we are interested in investigating duality in the context of robustness, let us state the (standard) dual to (I)'. So, introducing dual variables $\lambda_j, (j = 1, ...p)$, the dual to (I)' can be written as:

$$\min \quad \sum_j \lambda_j w^j = \sum_j \lambda_j \min_{b \in B} (u^j)^T b$$
$$\text{s.t:} \quad \sum_j \lambda_j F^T u^j \geq \gamma$$
$$\lambda_j \geq 0, \forall j = 1, ..., p$$

and since: $\{u/u = \sum \lambda_j u^j, \lambda_j \geq 0\} = \{u/G^T u \geq 0, u \geq 0\}$, this can be rewritten as:

$$\text{(DI)'} \quad \min_u \min_{b \in B} \quad u^T b$$
$$\text{s.t:} \quad F^T u \geq \gamma$$
$$G^T u \geq 0$$
$$u \geq 0$$

or , equivalently if we denote W denote the set of solutions to (DI)'

$$\min_{u \in W} \min_{b \in B} \{u^T b\} \tag{1}$$

An a priori different way of using duality in our context would be to take the (standard) LP dual to (I) for fixed b, and then to carry out robustness analysis with respect to the coefficients b of the objective in the dual, allowing b to take all possible values in B . The LP dual to (I) reads:

$$\text{(DI)} \quad \min u^T b$$
$$\text{s.t:} \quad F^T u \geq \gamma$$
$$G^T u \geq 0$$
$$u \geq 0$$

A natural robust version of (DI) consists in finding the dual solution u minimizing the value of $u^T b$ produced by the worst possible b, reads:

$$\min_{u \in W} \max_{b \in B} \{u^T b\} \tag{2}$$

which is to be contrasted with (1): indeed, it is seen that the robust version of the dual significantly differs from the dual of the robust version of the initial (primal) problem (I) because the function of u to be minimized is $\min_{b \in B} \{u^T b\}$ in one case, and $\max_{b \in B} \{u^T b\}$ in the other case.

It is worth observing that this structural difference between the two functions also implies a difference with respect to the practical solvability of the corresponding problems. The objective function in (2) is convex in u, making the robust version of (DI) efficiently solvable, whereas the objective in (1) is concave in u, making (DI)' and thus (by standard LP duality) (I)' too, difficult problems in the general case.

Remark 1: As already suggested above, the two-stage robust decision model proposed here is capable of producing less conservative solutions as compared with Soyster's model. The reason for this is that if, for a given uncertainty set B, we consider Soyster's model for problem (I), the problem to be solved is

$$\max_{y \in Y_s} \{\gamma^T y\}$$

where the set Y_s is defined as $\{y/y \geq 0$ and $\exists z \geq 0 : Gz \leq \underline{b} - Fy\}$ with $\underline{b}$ defined as:

$$\forall i, \underline{b}_i = \min_{b \in B} \{b_i\}$$

It is easily seen that $Y_s \subseteq Y = \{y/y \geq 0$ and $\forall b \in B, \exists z \geq 0 : Gz \leq b - Fy\}$ and cases where strict inclusion holds (leading to an improved robust optimal solution value over the optimal value of Soyster's model) can easily be found, as illustrated by the following example.

In this example we consider 3 variables $y \geq 0, z_1 \geq 0$ and $z_2 \geq 0$, and 3 constraints:

$$\begin{aligned} y - z_1 \leq & \ b_1 \\ y - z_2 \leq & \ b_2 \\ z_1 + z_2 \leq & \ b_3 \end{aligned}$$

and the uncertainty set B is taken as the set containing the two vectors $(1,0,1)^T$ and $(0,1,1)^T$. The objective function is to maximize y. It is easily checked that the set Y_S corresponding to Soyster's model is in this case the real interval $[0,1/2]$ leading to an optimal robust solution value 0.5. On the other hand, the set Y corresponding to our two-stage model is the real interval $[0,1]$ leading to the (less conservative) optimal robust solution value 1. Indeed, the value $y = 1$ is feasible in our model because, in case $b = (1,0,1)^T$ occurs, we can take $z_1 = 0$ and $z_2 = 1$; and, in case $b = (0,1,0)^T$ occurs, we can take $z_1 = 1$ and $z_2 = 0$. (Observe, as already pointed out above, that the value taken by the z variables indeed depends on which $b \in B$ is actually realized). Of course, this example does not rule out the possibility of having $Y = Y_S$ for some special instances. As will be seen in the next section, this possibility will arise in connection with the robust PERT scheduling problem, in the special case (referred to there as ***Case 1***) where the uncertainty set on the task durations is the cartesian product of a family of real intervals.

5 An application of the 2-stage robust LP model with RHS uncertainty: robust PERT scheduling

In this section we specialize the general two-stage robust LP decision model investigated above to robust PERT scheduling, with an uncertainty set D on the durations of the tasks, supposed to be given as a (finite or infinite) list of 'scenarios'. More precisely, we want to determine an earliest termination date which can be achieved for any realization of the task durations d in a given uncertainty set D.

5.1 Formulation as a 2-stage robust LP model

Consider a PERT network represented as a directed circuitless graph N in which the nodes correspond to tasks (the tasks are numbered $i = 1,2,..n$, the set of tasks is denoted I), and there is an arc (i,j) with length (duration) d_i whenever there is a precedence constraint stating that processing of task j should not start before completion of task i. The set of arcs is denoted U. We assume that node 1 has no immediate predecessor (it thus represents the initial task) and node n has no direct successor (it thus represents the terminal task of the project). Denoting $y_j (j = 1,..n)$ the starting date for each task j, and assuming first that the task durations d_i are exactly known, we want to minimize the total duration of the project while satisfying all precedence constraints, in other words:

$$\begin{aligned} \text{Maximize} \quad & -y_n \\ \text{s.t:} \quad & \\ & y_1 && = 0 \\ & y_i - y_j && \leq -d_i, \forall (i,j) \in U \end{aligned}$$

Indeed, it is easy to check that in the above, $y_1 = 0$ can be replaced by $y_1 \geq 0$, or equivalently $-y_1 \leq 0$, thus the problem can be rewritten:

$$\begin{aligned} \text{Maximize} \quad & -y_n \\ \text{s.t:} \quad & \\ & -y_1 && \leq 0 \\ & y_i - y_j && \leq -d_i, \forall (i,j) \in U \end{aligned}$$

This model is recognized as a special case of (I), the constraint matrix $[F, G]$ being formed by the transpose of the node-arc incidence matrix of N with an additional row involving variable y_1 only (with associated coefficient -1). F is reduced in this case to a single column (the column corresponding to node n in the transpose of the incidence matrix of N). The right handside vector b is the vector with coefficients equal to the opposite of the task durations (more specifically, the right handside coefficient for the constraint corresponding to arc $(i, j) \in U$ is equal to $-d_i$). Note that we do not state explicitly the nonnegativity conditions on y, since they are implied by the precedence constraints and nonnegativity of the d_i coefficients.

Thus the problem is cast in a form very similar to (I) the only difference being that the nonnegativity conditions on y and z are dropped. The consequence of this on the analysis of 3.2 is just that we have to consider the polyhedral cone $C' = \{u/u^T G = 0 \text{ and } u \geq 0\}$ instead of the polyhedral cone: $C = \{u/u^T G \geq 0 \text{ and } u \geq 0.\}$

Due to the special structure of the G matrix arising in the PERT scheduling problem, we have the following result:

Proposition 1: The extreme rays of the polyhedral cone C' are in 1-1 correspondence with the characteristic vectors of the various paths between node 1 and node n in N.

Proof: By observing that G^T is the node-arc incidence matrix of the graph N without the row associated with node n but with an extra column with coefficient -1 in the first row and all other coefficients 0, it is realized that u satisfying $G^T u = 0$ and $u \geq 0$ corresponds to a nonnegative flow between node 1 and node n in N with value equal to $u_{1,1}$, the component of u corresponding to the extra column with coefficient -1 in the first row and all other coefficients 0. Therefore the extreme rays of the cone C' correspond to the incidence vectors of the various paths connecting node 1 to node n in N. ∎

Let us denote $P = \{\pi^1, \pi^2, ..., \pi^K\}$ the set of all paths between 1 and n in N, $u^1, u^2, ..., u^K$ the corresponding characteristic vectors, the condition $(u^k)^T(b - Fy) \geq 0$ specializes to: $-\sum_{i \in \pi^k} d_i + y_n \geq 0$ (this is because, in that case: $(u^k)^T b = -\sum_{i \in \pi^k} d_i$ and $(u^k)^T F y = -y_n$). So the condition for feasibility is that for each path π^k : $y_n \geq \sum_{i \in \pi^k} d_i$.

In view of this, the robust PERT scheduling problem can be reformulated as:

$$\begin{aligned} \max \quad & -y_n \\ \text{s.t:} \quad & \\ & y_n \geq \sum_{i \in \pi^k} d_i, \forall \pi^k \in P, \forall d \in D \end{aligned}$$

where we recall that D denotes the uncertainty set for the task durations.

This problem therefore reduces to determining the path π^k maximizing, over the set P of all possible paths in N, the objective function:

$$\max_{d \in D} \{\sum_{i \in \pi^k} d_i\}$$

in other words we want to solve:

$$\max_{\pi \in P} \max_{d \in D} \{\sum_{i \in \pi} d_i\} \text{ (RPS)}$$

('Robust Pert Scheduling' problem)

Now, if we want to go further into the analysis of (RPS), we have to specify how the uncertainty set D is defined. Of course there are many possible ways for this; we content ourselves below to examine two among the most natural possible definitions, and show that, for each of them, the above robust optimization problem (RPS) can be efficiently solved .

Case 1: *D is a scaled ball w.r.t. the L_∞ norm.*

The first easy special case is when, for each task i, the duration d_i can take any value in a given real interval $[d_i^-, d_i^+]$ with $0 \le d_i^- \le d_i^+$. In this case D is the cartesian product: $[d_1^-, d_1^+] \times [d_2^-, d_2^+] \times ... \times [d_n^-, d_n^+]$, which may be viewed as a scaled ball w.r.t. the L_∞ norm (using component-wise scaling to have all intervals of equal width). It is easily seen that an optimal robust solution for problem (RPS) can be obtained in this case by looking at a longest path (critical path) in N when each of the tasks i is assigned the longest possible duration d_i^+.

As an illustration of the above, consider the following example with $n = 7$ tasks, where the graph of precedence constraints has the following arcs: $(1,2)$, $(1,3)$, $(2,3)$, $(2,5)$, $(2,6)$, $(3,4)$, $(3,7)$, $(4,5)$, $(5,7)$ and $(6,7)$. Thus task 2 cannot be started before completion of task 1, etc. Also note that the tasks are numbered according to a topological ordering of the graph, since there is no arc (i,j) with $i > j$. The associated intervals $[d_i^-, d_i^+]$ for the durations of the tasks are the following:

Task 1	Task 2	Task 3	Task 4	Task 5	Task 6
$[2,4]$	$[4,8]$	$[3,6]$	$[4,8]$	$[4,8]$	$[8,16]$

Task 7 is not shown in the above table because it is a dummy task (of duration 0, without uncertainty) representing the end of the schedule. It is easy to see that in this example, the optimal solution to the (RPS) problem has duration 34 and corresponds to the critical path $(1,2,3,4,5,7)$. Indeed 34 is the earliest achievable termination date if we require that the schedule remains feasible for any possible choice of the task durations in the cartesian product $[d_1^-, d_1^+] \times [d_2^-, d_2^+] \times ... \times [d_6^-, d_6^+]$. This corresponds to the situation where the duration of each task i is d_i^+.

Case 2: *D is a scaled Hamming ball of bounded radius Γ.*

Here again we assume that, for each task i, the duration d_i can take any value in a given real interval $[d_i, d_i + \Delta_i]$ with $d_i \ge 0$. d_i is called the nominal value of the duration for task i, $d_i + \Delta_i$ being the possible extreme (or worst-case) value for the task duration. As is actually the case in many practical applications, it is unlikely that all tasks simultaneously take on worst-case values. To take this observation into account, we will impose an upper bound Γ on the number of task durations which are allowed to take on a worst-case value, given that all task durations which do not take on a worst-case value are assumed to be equal to their nominal value. More formally, associating with each task i a 0-1 integer variable u_i , the uncertainty set D corresponding to this definition is:

$$D = \{\theta = (\theta)_{i=1,\dots,n} / \theta_i = d_i + \Delta_i u_i (i = 1, ..., n)$$

such that: $\sum_{i=1}^{n} u_i \le \Gamma, u_i \in \{0,1\}, \forall i\}$.

As can be seen from the above definition, in the special case where all Δ_i are equal to 1, D is recognized as the Hamming ball of radius Γ centered at $d = (d_i), i = 1, ..n$ (in other words, $\theta - d$ can be any 0-1 vector with at most Γ components equal to 1). When the Δ_i's take on arbitrary positive values, the Hamming ball structure is still present after applying scaling to each component i with respect to the corresponding Δ_i value.

We note here that, in spite of the fact that the definition above is close in spirit to the concept of uncertainty suggested by Bertsimas and Sim (2003, 2004), our model is fairly different from the one studied by these authors, since they restrict themselves to rowwise uncertainty, whereas in our robust PERT scheduling problem, we have uncertainty on the RHS only (a special case of columnwise uncertainty). For more detailed discussion of this issue, see 4.2 below.

We now show that, with the above definition of the uncertainty set D, problem (RPS) can be efficiently solved via a dynamic programming recursion. To that aim, we will consider the problem with parameter Γ as only one representative of the class of problems (RPS$[i,k]$) for i running from 1 to n (the number of tasks) and k running from 0 to Γ. More precisely, assuming that the tasks are numbered according to a topological ordering of the circuitless graph N, (RPS$[i,k]$) consists of the robust PERT scheduling subproblem corresponding to the subset of tasks 1, 2,...,i, the durations of at most k of which are allowed to take on their worst-case values. The case k=0 (no deviation allowed) corresponds to the usual PERT scheduling problem in terms of the nominal values for the task durations. We denote $v^*[i,k]$ the optimal objective function value for problem (RPS$[i,k]$), and for any task i, we denote $Pred[i]$ the set of tasks j such that (j,i) is an arc of N (the set of direct predecessors of node i). Bellman's optimality principle then leads to the following dynamic programming recursion:

$$\forall i \in [1,n], \forall k = 0,1,...\Gamma :$$
$$v^*[i,k] = \max_{j \in Pred[i]} \max\{v^*[j,k] + d_j;$$
$$v^*[j,k-1] + d_j + \Delta_j\} \qquad (3)$$

The optimal value of the robust PERT scheduling problem we are interested in is then: $v^*[n,\Gamma]$. The rationale behind Eq. (3) can easily be explained as follows. Consider the set of all paths from 1 to $j \in Pred[i]$. The duration of arc (j,i) has nominal value d_j and worst-case value $d_j + \Delta_j$. The maximum duration of a path from 1 to i through j with at most k tasks allowed to deviate from their nominal values can be obtained: either by allowing for at most k deviations on the subset of tasks $\{1,2,..j\}$ and taking the nominal duration for arc (j,i); or by allowing at most $k-1$ deviations on the subset of tasks $\{1,2,..j\}$ and taking the worst-case duration for arc (j,i). Thus the optimal value for node i via node j is the maximum value among these two alternatives, and the optimal value for node i is the maximum taken on the set of all direct predecessors of i. Obviously, solving the recursion (3) is achieved in polynomial time $O(m \times n)$, where m is the number of arcs and n the number of nodes of the PERT network; more precisely the complexity is $O(m \times \Gamma)$ where Γ, the parameter defining the uncertainty set, is at most n, the number of tasks (but often significantly smaller than n in practical applications).

Let us illustrate the above on the same 7 task example as the one considered to illustrate case 1. We thus consider the same intervals for the task durations, the lower bound of each interval representing the nominal task duration, and the upper bound representing the worst-case duration. For $\Gamma = 3$, application of the recursion (3) leads to the $v^*[i,k]$ values shown in the following table.

	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$k=0$	2	6	9	13	6	17
$k=1$	4	10	13	17	10	22
$k=2$	4	12	16	21	12	26
$k=3$	4	12	18	24	12	29

For instance the value $v^*[4,2] = 16$ corresponds to the path $(1,2,3,4)$ with 3 arcs of nominal durations 2, 4 and 3 respectively. If, in this path, two arcs out of three are allowed to take on their worst-case durations, the worst case (Max) is obtained when task 2 has duration 8 and task 3 has duration 6 (task 1 keeping its nominal duration 2), the resulting length of the path being: 8 + 6 + 2 = 16. Let us also illustrate how the recursion (3) works for computing e.g. $v^*[7,2]$ and $v^*[7,3]$. We have:

$$\begin{aligned} v^*[7,2] =& \max\{v^*[3,2]+3; v^*[3,1]+6; \\ & v^*[5,2]+4; v^*[5,2]+4; v^*[5,1]+8; \\ & v^*[6,2]+8; v^*[6,1]+16\} \\ =& \max\{15,16,25,25,20,26\} = 26. \end{aligned}$$

The maximum above is obtained for $j = 6$ and the corresponding optimal path is $(1,2,6,7)$.

Similarly we have:

$$v^*[7,3] = \max\{v^*[3,3]+3; v^*[3,2]+6; \\ v^*[5,3]+4; v^*[5,2]+8; \\ v^*[6,3]+8; v^*[6,2]+16\} \\ = \max\{15,18,28,29,20,28\} = 29.$$

The maximum above is obtained for $j = 5$ and the corresponding optimal path is $(1,2,3,4,5,7)$.

It is thus seen that, depending on the choice of the control parameter Γ, various optimal paths are obtained which, of course, may differ from the optimal solution to the non-robust PERT scheduling problem (considering only the nominal values for the task durations). Also, observe that the value $v^*[7,3] = 29$ corresponds to a less conservative robust situation as compared with the one obtained in case 1 above.

5.2 Differences with Bertsimas and Sim's approach

We now turn to show that, in spite of the similarity in the definition of the uncertainty sets, the robust version of the PERT scheduling problem investigated here is essentially different from the model proposed by Bertsimas and Sim [3] for the robust version of the shortest path problem. From an abstract point-of-view, the difference basically stems from the fact that, in our case, we are faced with a LP problem with uncertainty on the right handside, whereas Bertsimas and Sim address a LP problem with uncertainty on the cost coefficients. However, to further understand the source of this difference, we show below which difficulties would arise if we wanted to apply the Bertsimas-Sim approach to the robust longest (critical) path problem on a directed circuitless graph G.

Following these authors, the robust shortest s-t path problem in G with uncertainty parameter Γ (assuming $\Gamma \in \mathbb{N}$) is formulated as:

$$\text{(RSP)} \quad \min_{x \in X}\{\sum_{(i,j)\in U} c_{i,j}x_{i,j} + \max_{S\subseteq U, |S|\leq\Gamma} \sum_{(i,j)\in S} \Delta_{i,j}x_{i,j}\}$$

where X denotes the set of incidence vectors of all s-t paths in G; $c_{i,j}$ denotes the nominal cost of arc (i,j) and $c_{i,j}+\Delta_{i,j}$ is the worst-case cost of arc (i,j). After transformation of (RSP) using the duality theorem to convert the second term in the brackets into a minimization, the problem is reformulated as a standard LP , the solution of which reduces to $m+1$ aplications of a standard shortest path algorithm. Observe that one of the reasons for all the above to work so nicely is that the second term in the brackets, as a function of x, is convex in x , since it is the pointwise maximum of a finite number of linear functions.

The above approach is still valid if, instead of looking for an optimum robust minimum cost path, we were looking for an optimum robust maximum benefit path: $c_{i,j} > 0$ being interpreted as a reward associated with the use of arc (i,j), the effect of uncertainty being to reduce the nominal reward $c_{i,j}$ by the amount $\Delta_{i,j}$. The problem would then take the form:

$$\max_{x \in X}\{\sum_{(i,j)\in U} c_{i,j}x_{i,j} - \max_{S\subseteq U, |S|\leq\Gamma} \sum_{(i,j)\in S} \Delta_{i,j}x_{i,j}\}$$

which is essentially analogous to the above robust minimum cost path, up to a change in the signs of the coefficients in the objective (still assuming, of course, that the graph G under consideration is circuitless). In particular, we note that the function to be maximized is concave, so we still have a convex optimization problem.

By contrast, the robust PERT scheduling problem addressed in the present paper is formulated as:

$$\max_{x \in X}\{\sum_{(i,j)\in U} c_{i,j}x_{i,j} + \max_{S\subseteq U, |S|\leq\Gamma} \sum_{(i,j)\in S} \Delta_{i,j}x_{i,j}\}$$

(with $c_{i,j} > 0$ and $\Delta_{i,j} > 0$).

It is then readily observed that this problem consists in maximizing a convex function of x on $\{0,1\}^m$, and it is well-known that this cannot be simply reduced to ordinary linear programming as is the case for Bertsimas and Sim's approach. Thus robust PERT scheduling may be viewed a typical illustration of the big differences between models featuring rowwise uncertainty and models featuring columnwise uncertainty in robust Linear Programming.

6 Conclusions

In this paper, various Robust Linear Programming problems have been investigated, and the question of whether LP duality can still be used to help in solving such problems has been addressed and answered negatively. Among the problems considered, Robust Linear Programming problems with uncertainty in the right hand-sides only, have been recognized as an interesting sub-class of problems, for which the solution techniques should not confine themselves to the classical approach proposed by Soyster (1979). In this respect we have been lead to propose a new class of robust LP models referred to here as 'Two-Stage Robust Decision Models' which can be expected to lead to less 'conservative' optimal robust solutions than those usually obtained from Soyster's model. In order to show the practical usefulness of this 2-Stage model, a specialization to robust PERT scheduling has been discussed, leading , under two natural ways of defining the uncertainty set w.r.t. the task durations, to efficient solution methods. Also some fundamental difference between our approach to robust PERT scheduling and the one proposed by Bertsimas and Sim [3] in the context of the robust shortest path problem has been pointed out. We think that many other possible applications of this 2-Stage robust modeling approach would deserve further investigations, for instance in dynamic inventory management, optimal resource allocation problems, telecommunication problems, etc.

References

[1] BEN-TAL A., NEMIROVSKI A. (1998) Robust Convex Optimization, Math. Oper. Res., 23, pp. 769-805.

[2] BEN-TAL A., NEMIROVSKI A. (2000) Robust Solution of Linear Programming Problems Contaminated with Uncertain Data, Math. Prog., 88, pp. 411-424.

[3] BERTSIMAS D., SIM M. (2003) Robust Discrete Optimization and Network Flows, Math. Prog. B, 98, pp. 49-71.

[4] BERTSIMAS D., SIM M. (2004) The Price of Robustness, Oper. Res. 52, 1, pp. 35-53.

[5] CHINNTECH J.W., RAMADAN K. (2000) Linear Programming with Interval Coefficients, The Journal of the Operational Research Society, 51, 2, pp. 209-220.

[6] INUIGUCHI M., SAKAWA M. (1995) Minimax Regret Solution to Linear Programming Problems with an Interval Objective Function, E.J.O.R., 86, pp. 526-536.

[7] KOUVELIS P., YU G. (1997) Robust Discrete Optimization and its Applications, Kluwer Acad. Publ., Boston.

[8] MULVEY J. M., VANDERBEI R. J., ZENIOS S.A. (1995) Robust Optimization of Large Scale Systems, Oper. Res. 43,2, pp. 264-281.

[9] SOYSTER A.L.(1973) Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming, Oper. Res. 21, pp. 1154-1157.

[10] SOYSTER A. L. (1979) Inexact Linear Programming with Generalized Resource Sets, EJOR, 3, pp. 316-321.